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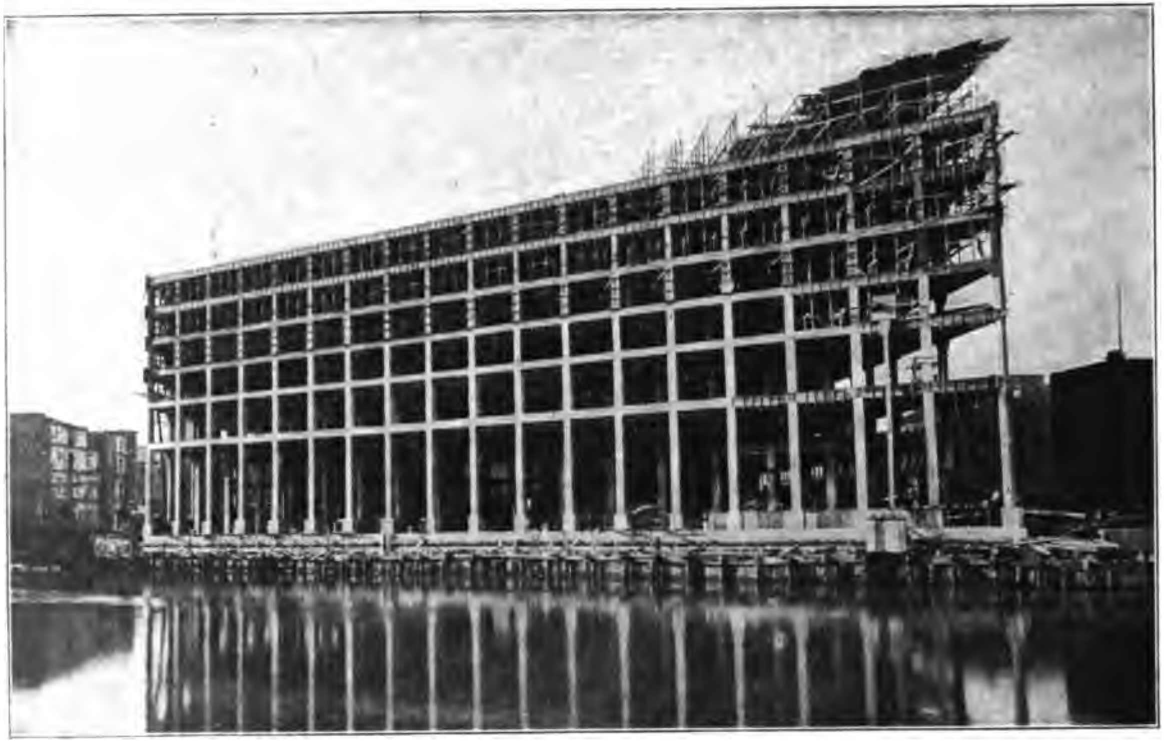

BOSTWICK-BRAUN COMPANY BUILDING, TOLEDO, OHIO

A. Bentley & Sons, Contractors "Mushroom System"

CONCRETE-STEEL CONSTRUCTION

PART I—BUILDINGS

A TREATISE UPON THE ELEMENTARY PRINCIPLES OF DESIGN
AND EXECUTION OF REINFORCED CONCRETE
WORK IN BUILDINGS

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SECOND EDITION, REWRITTEN AND EMBODYING THE MOST
ADVANCED THEORETICAL DEVELOPMENTS

MINNEAPOLIS

1919

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PREFACE

According to Mörsch, practice in reinforced concrete construction has been far ahead of theory, and the principal point of controversy in this theory has been whether the tensile strength of concrete can be considered as other than negligible in a concrete structure under working and test loads. Experience influenced early practical builders against its inclusion, and the practice of reinforced concrete construction as governed by building code regulations today requires that the steel shall be assumed to furnish all the direct tensile resistance.

But the theory of reinforced concrete as advanced in the discussion of the Report of the Joint Committee, Trans. Am. Soc. C. E., Vol. LXXXII, p. 1547-1562 to account for results observed in tests is quite another matter. Here the tensile strength of concrete, which practical builders and all city building codes hold should be considered as a merely negligible resistance is represented as actually resisting three fourths of the statical applied bending moment in column supported flat slabs under test loads.

This attempt to reconcile experiment with false theory, and thus establish an indefensible theory recalls the reasoning of the medieval philosophers, who treated theory and practice as denoting conflicting and mutually inconsistent ideas, under the fallacy of a double system of natural laws, one discoverable by contemplation as the product of noble and liberal art, the other practical and discoverable by experience.

The practical builder, assumes the responsibility for the strength and capacity of the structure he builds, which for success requires exact knowledge wholly unnecessary in the formulation of a plausible theory.

The departure of the early concrete builder from old forms of construction required the development of a new theoretical treatment based upon a broader application of the ordinary principles of equilibrium.

The pioneer English treatise on the principles of mechanics of equilibrium as applied to engineering structures is that of Rankine published in 1858 and devoted largely to the flexure of members in cylindrical curvature in which no attention was given to combined twisting and bending as found in the flexure of flat plate floors.

Later engineering treatises have followed the lead of this eminent theoretical writer, modifying and enlarging in relatively minor particulars the propositions treated by him. No broad investigation of the flexure of plates in double curvature along the lines that have been partly developed by the eminent physicists Thomson and Tait in their treatise on Natural Philosophy appears to any considerable extent in the engineering text books of today. Accordingly, although the fundamental principles of bending in synclastic and anticlastic curvature were taught to some extent half a century ago, they are today almost unknown to the average engineer and are given scant attention in the attempted solution of the problems of reinforced concrete construction in which these principles are called into play.

The mathematical treatment of the balance of statical moment by combined twisting and bending moments has been so incompletely understood that the fundamental principles underlying it have after ten years consideration by a Joint Committee of the Am. Soc. C. E. and others composed in part of representative men from our engineering colleges been wholly disregarded and their applicability denied. The fundamental theorems setting forth the principles governing the relation of the internal work of deformation and the external work of the applied loads have also been heretofore disregarded in obtaining a basis for the formulation of rules for the design of reinforced concrete structures.

The effort of the present authors is to elucidate the harmony existing between correct theory and scientific practice and disclose the error of commonly accepted* methods of computation of applied bending moments by showing how unwarranted is the assumption that the statical moment about any section in a plate is all pure bending moment, instead of being a combination of applied bending and twisting moments as treated by Thomson and Tait, in their great treatise on Natural Philosophy published at Oxford University in 1867, see sections 627-648.

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*These erroneous methods appear to the junior author to be the result of a faulty method of instruction in our engineering schools, under which the student learns to recite mechanical principles rather than to apply them to a useful end. This viewpoint resulted from the study of testimony in patent suits, in which a professor of engineering testified that the mode of operation of a flat slab would be the same before the concrete cracked whether the steel were in the top or bottom, notwithstanding that he was able to correctly recite the principle of rigidities and law of least work. The further testimony of a university research fellow, who did not believe there could be any relation between the external work of the load and the internal work of deformation because he had never been able to measure this work with an extensometer, strengthens this conclusion.

The discussion of the analysis of the column and flat slab problem furnished by the authors in 1914 by attacking it thru analogy to a plate of some kind of homogeneous material, brought to light so many misconceptions of fundamental principles that a thoro treatment of elementary principles seemed essential if the subject matter were ever to become clearly and generally understood.

In presenting a correct analysis of bending and twisting moments, an effort has been made to avoid higher mathematics and to simplify the subject matter so far as possible.

In doing this the analysis here given is founded, as all correct analysis of applied bending and twisting moments in loaded plates must necessarily be founded, upon the manner of distribution of the vertical shears that transmit the loading to the supports. But the actual distribution of vertical shears in plates has apparently never heretofore occupied the attention of any who have treated reinforced flat slabs. It is a matter, however, that has been investigated in detail in the following pages in which it is shown that the distribution is fully determined when the lines of zero shear are known. The lines of zero shear have been figured for the various kinds of support. From these the precise distribution of shear intensities becomes known, which furnishes the basis for a correct determination of the bending and twisting moments by reason of the necessary relationship they bear to the shears that produce them. In case of plates and slabs bending moments are not identical with statical moments as they are in case of beams.

A fundamental misconception of the equilibrium of an element in shear becomes apparent in the investigation herein presented. Strangely enough all mathematical writers on the subject of elasticity heretofore have only considered a special case of the general problem and have assumed a special solution to be a general solution with the result that the increased shearing resistance of the continuous beam as contrasted with the simple beam has never heretofore been satisfactorily explained.

Investigation of the nature and equilibrium of residual strains has not been heretofore attempted. The combined effect of polar and transverse moments of inertia in determining the strength of members in compression is here suggested while the apportionment of the applied moment at mid span and over the supports thru the relation of the moment area to moment of inertia is given thorough consideration for the first time. The underlying idea of the treatment throughout the work is that it is necessary for the reader to understand elementary principles before he becomes competent to grasp the more intricate and complex discussion of practical details, and the development of these principles has been given close attention as a preliminary study for a more extended discussion of practical work and details later on.

The evil influence of the supposed inconsistency of theory and practice is characterized by Rankine as a thing that opposes communication of ideas between men of science and men of practice, and one which leads scientific men to employ much time and mental exertion upon ingenious mathematical exercises which would be better bestowed on questions connected with the art, and further that it leads them to state results of important investigations on practical subjects in forms too obstruse for ordinary use so that the benefit which should be derived from their application is lost to the public for years and valuable practical principles which might have been investigated by reasoning are left to be discovered by slow and costly experience.

But it is in the methods of engineering, as Rankine says, that the influence of this great fallacy is most conspicuous.

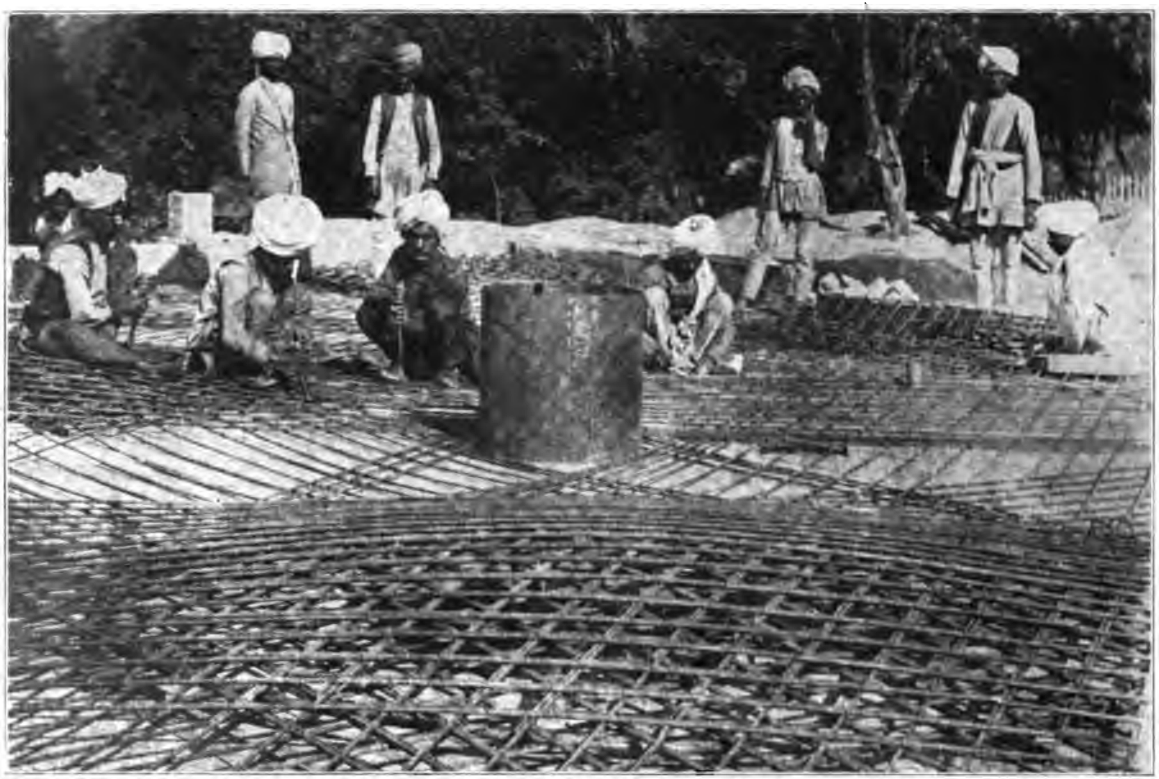
> "Of that scientifically practical skill which produces the greatest effect with the least expenditure of material and work instances are comparatively rare. In too many cases we see the strength and stability which should be given by skillful arrangements of the parts of a structure supplied by means of clumsy massiveness and lavish expenditure of material, labor and money, and this evil is increased by a preversion of public tastes which causes engineering works to be admired not in proportion to their fitness for their purpose, or the skill in attaining that fitness, but in proportion to their size and cost."

The best efforts of the authors have been directed toward the elimination of these evils, in the field of reinforced concrete by the scientific application of the fundamental principles of mechanics to the elementary types of concrete building construction that the enormous fire loss of half a billion dollars per year in this country may be eliminated. In case of uncertainty as to theoretical principles to be applied, the designer for safety employs material in such lavish manner as to render the cost of the fireproof building prohibitive when in fact noncombustible material, such as reinforced concrete, scientifically designed, may cost no more than the inflamable timber structure.

Before the success of Turner in building flat slabs the theoretical idea of a rectangular mat of crossed rods embedded in concrete was that each set of parallel rods could resist only longitudinally and separately in proportion to its cross section and that any action greater than this resistance must be credited to the tensile resistance of the matrix which is admittedly not dependable.

Were it possible to make a single sheet of metal of the mat, the sheet would have twice the cross section and furnish twice the resistance in each direction which is furnished by the single set of rods in that direction, and the discovery of Turner was that the matrix has an effect to make the sets of crossed rods act together with an efficiency comparable to that of a sheet. This action is brought about by the bond shear resistance furnished by the embedment and is a matter not of assumption or to be determined intuitively, but a matter which is to be determined by experiment. Such action, as shown by experiment, causes the plate with a fine grained mat to furnish even more effective resistance in proportion to the steel employed than would be furnished by the action of a homogeneous sheet of steel in the tension zone and the mechanical principles known to apply to such a homogeneous sheet are set forth in the accompanying discussion together with the peculiar modifications of this action which takes place in a non-homogeneous plate a well as the conditions under which it is and is not efficient.

The Authors



GARRISSON BAKERY FOR BRITISH GOVERNMENT, JULUNDER DISTRICT, DALHOUSIE, INDIA
Native laborers placing reinforcement steel—Mushroom System

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UNIV. OF CALIFORNIA

CONCRETE STEEL CONSTRUCTION

CHAPTER 1.

1. INTRODUCTORY

The history of structural engineering as a science dates from the early part only of the last century. The progress made has been remarkable indeed, and the materials mainly used have varied during well-defined periods. Up to 1860, timber, wrought and cast iron were mainly used; from 1860 on wrought iron with some cast iron was generally employed in bridges and other engineering structures; from 1890 to the present time steel has replaced wrought iron; and while, for long-span bridges, it will perhaps be some time before a more suitable metal is found, yet for short spans, buildings, warehouses and the like, the enterprise of the manufacturers of Portland cement has placed at the disposal of the engineer a new material, reliable, if properly handled, and of reasonable cost, which bids fair to *largely supplant* steel in the construction of minor engineering works. Indeed today, a warehouse designed for a capacity of 400 pounds per square foot of floor, columns 16 to 24 feet centers, can be built more cheaply of reinforced concrete than of wood frame and floor, with similar brick walls. Where the strength required is less, timber at the present rate is slightly cheaper, since the cost of centering is the same for light as for heavy construction. Still, the difference is so slight that, considering the saving in insurance, owners will shortly be convinced that they cannot afford to continue the construction of fire traps if they are to realize the maximum profits on their investments.

The strength of Portland concrete in compression is equal to that of good building stone, with the advantage that it can be placed in monolithic masses. Its tensile strength, like stone, is greatly inferior to that in compression. Concrete yields but little, the stretch being confined to the weak section. When, however, steel is embedded in the concrete and properly disseminated thru it, the deformation or stretch is distributed greatly by the metal.

The conditions leading to the combination of concrete and steel in a beam or girder are these: Concrete is an excellent and trustworthy material for compression and steel for tension. Hence steel should be distributed in such manner as to carry the tensile stresses of the chord and web. To do this economically we can reason from analogy of a truss or beam. The further from the neutral axis the more effective the steel section, hence the reinforcement for tensile chord stress should be at the bottom of the beam or as close to it as satisfactory protection against heat by fire will permit. Now the beams in a building are of constant section, and since a continuous beam is stiffer and stronger than a beam of the same section discontinuous over supports, the ideal concrete-steel beam should be continuous and the top flange reinforced over supports.

As Mörsch states in his treatise:—

> "Practice has been far ahead of theory. The principal question in controversy has been whether the tensile strength of concrete in bending should be considered. Among practical builders this was decided at the start, and decided against its inclusion, because absolutely no attention is paid to it and the steel is stressed to the safe limit. The tensile strength of the concrete is entirely ignored. On this assumption is based the first method of the theoretical computation of slabs devised by Koenen in Berlin in 1866 and his method has been used by the majority ever since."

2. Historical

Steel and concrete as a combination of materials for engineering structures is much older than is generally supposed. Professor Barbour of the University of Nebraska, in a very interesting lecture delivered to the Cement Users of the State of Nebraska, described certain flat arches discovered in the ruins of ancient Rome, which were for a long time a puzzle to engineers and architects until it was found that they were tied together and the thrust in large part resisted by iron tie rods. So far as known, little was done, however, in the way of combining the old Roman concrete with iron, except in the isolated instance just cited, and it was not until 1855 that iron was combined with concrete in a manner similar to that in which it is utilized at the present time.

At the Paris Exposition of 1855, Lambot exhibited a boat made of reinforced concrete, while François Coignet is credited with having built floors and pipes, in the construction of which he had combined steel and concrete to some extent.

In 1867, Scott, a Lieutenant Colonel of Engineers of the British Army, took out a British patent on concrete floor slabs reinforced in one direction, and also in two directions, and in some of the drawings in this early patent woven fabric is found combined with rods.

In France, Joseph Monier took out patents about the same time, and to him, perhaps more than to any other person, is to be given credit for the commercial introduction of reinforced concrete on a large scale. His first use of this type of construction was to fabricate large plant tubs which he found more durable than those of wood, and more readily transported than those of cement without reinforcement. In 1867 he took out his first French patent, which he soon followed with a number of others on reservoirs, floors, and straight and arched beams in combination with the floors, etc. In 1884 the Monier patents were purchased by the firms of Freitag and Heidschuch in Newstad-on-the-Haardt and Martinstein and Josseaux in Offenbach-on-the-Main. Later the patent rights in Germany were sold to Engineer Wayss, under whose supervision tests were made in Berlin, the results of which were published in 1887, and on the basis of these experiments Wayss succeeded in introducing the Monier system into many structures.

The Scott patent is especially significant since the specification states that tie rods and hoop iron were to take the tensile strains and concrete the compressive stresses, showing clearly that Scott understood the basic principle of reinforced concrete. Little was done, however, by the earlier inventors in developing a working theory of design.

Herr Wayss conducted certain tests in Berlin and published his results in a pamphlet entitled "Das System Monier, Eisengerippe mit Zementumhullung." In this pamphlet Wayss expressed the opinion that the steel must be placed where the tensile stresses occurred. The tests were witnessed by government officials as well as by private engineers and architects. Government Architect Koenen, afterwards Director of the Actiengesellschaft fur Beton-und Monierbauten, in Berlin, was commissioned by Wyass to work out methods of computation from these tests, which were published in the volume of the "Zentralblätter der Bauverwaltung" for 1886.

Mörsch, Concrete Steel Construction, 1907 thus comments on the introduction of reinforced concrete at this period (1886).

> "Commencing at that time (1886) a theoretical foundation was evolved, according to which the design of reinforced concrete work would be effected, and through these preliminary labors, this method of construction was extensively adopted in Germany and Austria. A turning point in its development was the International Exposition in Paris, in 1900, and the report by von Emperger, published at that time in regard to the position which the subject occupied.
>
> Because of the scientific investigation of reinforced concrete during the past few years, it has made rapid progress in Germany.

> It was specially promoted by the publication, in 1904, through the cooperation of experts and practical men of the "Leitsätze" of the Verbands Deutscher Architekten- und Ingenieurvereine and the Deutschen Betonvereins as well as by the Regulations of the Prussian Government, which abolished many restrictive rules, cleared the way, and inspired in the widest circles confidence in the new method of building."

In 1870 Phillip Brannon made what appears to have been the first application for an English patent on reinforced concrete piles. The patent was granted in 1871, showing reinforced concrete piles with longitudinal reinforcement of angle irons united by bars riveted across them, the whole being wound spirally with wire.

Thaddeus Hyatt between the years 1873 and 1881 took out between thirty and forty different patents relating to reinforced concrete work, pavement lights, floors and slabs. It does not appear, however, that Hyatt made a success of concrete construction commercially altho he did make a success of his paving lights. Hyatt regarded a reinforced concrete beam as one corresponding to a steel beam, and he considered the rods as equivalent to the bottom flange and the concrete as the top flange, assuming the neutral axis at mid depth of the beam.

A most important patent was granted to Hyatt in 1874, in England, No. 1715, in which is disclosed spiral and vertical reinforcement for columns, which strangely enough was not appreciated until attention had been called to this type of column by its reinvention at a later date by Considère, to whom the engineering profession is indebted for the attention which he directed to it by the valuable tests carried out by him.

During this time there was considerable activity in the United States. E. L. Ransome was building reinforced concrete warehouses as early as 1884, and patented in the United States a twisted bar reinforcement.

In 1883 John F. Golding secured an American patent for expanded metal which was employed as lathing for plaster in lighter gages, and as reinforcement for concrete slabs with larger mesh and a No. 10 or heavier gage.

In Cassell's Reinforced Concrete, published in 1913, the following sketch is given of the work of the early pioneers of the art before 1900, Edmond Coignet and François Hennebique:—

> "The former of whom, by applying the known principles of mechanics, evolved a system of calculation that has proved remarkably truthful, and the latter of whom, basing his methods of calculation upon results obtained in practice, has also made extremely important contributions to the technical consideration

> of the subject. Coignet as the scientific investigator, and Hennebique as commercial organizer, are properly regarded as 'the pioneers of the modern evolution in the art of building.' The story has often been told of the opposition which Coignet had to fight in getting the masonry of the proposed new system of main drainage in Paris in 1892 replaced by reinforced concrete. He promised a large saving of money and of time required for construction, and his system, which was finally adopted, was carried out with complete success. Hennebique, having organized a technical staff and licensed a large number of the most influential contractors to work his system, was able to secure between the years 1892 and 1899 work to the total value of two million sterling, representing three thousand constructions, among the most remarkable of these being the bridge of Chatellerault, 460 ft. long, comprising three arches, two of 133 ft. span and one of 167 ft..
>
> Hennebique's first patent dates from 1892, (British patent, No. 14,530), and in this he demonstrates the utility of stirrups to reinforce beams against shear, in which matters he had to an extent been anticipated by Hyatt in 1877 and Meyenberg in 1891. In 1897 Hennebique introduced cranked-up rods, and placed these one above the other, so as to reduce the width of the beam, following (to some extent) the lines laid down by Hyatt in 1877 and F. G. Edwards in 1892, in which latter year M. Koenen and G. A. Wayss, of Germany, patented in England a method of floor construction with rods cranked-up at the point of contraflexure, "the parts in tension being strengthened by roughened or serrated metal rods or strips embedded in the structure."

We have noted the early work of Ransome in the United States. In his work parallel joists about 3 inches wide, spaced three feet centers were frequently used, while narrow intersecting ribs about ten feet centers in two directions surmounted by a thin slab were also employed.

From 1890 to 1900 while cement was relatively high in price, reduction of mass at increased expense in form work was to be expected.

The expanded metal companies introduced a large amount of short span concrete floors on structural steel frames six to ten feet center to center of beams, competing with hollow tile arches then more commonly employed for fireproof construction.

The Roebling Company, manufacturers of wire, were at the same time putting in a large amount of short span arches and slabs reinforced with wire fabric and rods.

From 1900 to date, reinforced concrete building construction has increased with wonderful rapidity, encouraged by the enterprise of the manufacturers of Portland cement in placing at the disposal of the constructor a reliable and uniform product at so low a cost that a most powerful impetus was thus furnished for the more complete development of the commercial possibilities of reinforced concrete in building construction.

The skepticism of building departments and the natural antagonism of tile interests forced the advocates of concrete construction to make innumerable tests with the final result of creating confidence in the construction when properly designed and executed.

During the six years subsequent to 1900—a short period of time but an epoch from the standpoint of progress in concrete construction—numerous beam theories were proposed and discussed.

Engineering opinion gradually crystallized in the adoption of a modification of the common theory of flexure and the assumption of the linear law of distribution of stress for purposes of computation of beams and slabs, and this opinion has been embodied in nearly all building codes in cities throughout the United States and Canada.

In 1904 a Joint Committee of the American Association of Cement Manufacturers, American Society for Testing Materials, American Society of Civil Engineers, American Railway Engineering and Maintenance of Way Association, was appointed to investigate and report on concrete and reinforced concrete. After eight years, a report was rendered, for which see Eng. News. Feb. 6, 1913.

*The report specifically states that it does not go into all types of construction or all the applications to which concrete and reinforced concrete may be put, * * * * * it is not a specification but may be used as a basis for specifications."

Treatment of natural types of reinforced concrete is lacking in this report. The treatment of beams, however, embodies the crystallized opinion above referred to and will be referred to more at length later.

That a theoretical treatment of concrete after the manner of structural iron work, although making allowance for the elastic properties of the two materials, is unsatisfactory was early recognized by practical men and also by some theoretical writers.

Marsh in his treatise on Reinforced Concrete, Edition of 1905, Part V. p. 209, makes the following remarks on this subject:

> "When properly combined with metal, concrete appears to gain properties which do not exist in the material when by itself, and although much has been done by various experimenters in recent years to increase our knowledge on the subject of the elastic behavior of reinforced concrete we are still very far from having a true perception of the characteristics of the composite material.
>
> "It may be that we are wrong from the commencement in attempting to treat it after the manner of structural iron work and that although the proper allowances for the elastic properties of the dual material is an advancement on the empirical formula at first employed and used by many constructors at the present

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*Since this was written a final report erroneously treating flat plates has been rendered.

> time, yet we may be entirely wrong in our method of treatment.
>
> "The molecular theory, i. e. the prevention of molecular deformation by supplying resistances of the reverse kind to the stresses on small particles, may prove to be the true method of treatment for a composite material such as concrete and metal. This theory is the basis of the Cottançin construction which certainly produces good results and very light structures, and M. Considère's latest researches on the subject of hooped concrete are somewhat on these lines."

In this statement of Marsh there is some idea of a possible new principle of action; but unfortunately he was unable to form and express any conception of how this might operate in accordance with mechanical principles in a manner which would be of benefit to the industry at large, or aid in the discussion of the stresses operating in reinforced concrete.

The practical constructor has an advantage over the theorist in this respect; having observed the results obtained by a new principle he immediately profits by it by taking advantage of the results through application of the principles of simple proportion leaving the explanation as a matter of academic interest to follow in the wake of his practical accomplishment. In this simple manner two thousand structures of the Mushroom type have been erected and tested for strength and deflection before a comprehensive scientific explanation of the mode of operation was forthcoming.

The constructor wants the least theory possible and that the simplest to meet the specific requirements of the work he has in hand. The scientist with broader and more comprehensive vision sees in the specific performance of the builder only a special case to be treated in conjunction with the species to which it belongs. That the latter treatment is by far the more difficult—and when correctly carried out more valuable and satisfactory—compensates for the almost inevitable position that the scientist occupies in a new art, following in the wake of the practical constructor, whose simple needs require special rather than general solutions of the problems at hand.

As the authors now view the art broad general solutions of the problems in reinforced concrete are in order and the character and nature of the new properties added to concrete by the dissemination of steel through it brought out in the following pages it is hoped may harmonize many differences in engineering opinion existing at the present time relative to the more advanced forms of construction.

In the discussion of concrete-steel construction, we must consider, first, the action of the concrete with the steel, the function of each in the combination, the problems presented by beams, slabs, and columns separately, and finally the mix of the concrete and questions of cost in convenient placing of the reinforcement.

Before taking up these points in detail it would seem in order, however, to turn our attention to the concrete, and the materials entering into it, their characteristics, value and fitness and the proper proportions to use.

3. Materials

Portland Cement only should be used in a reinforced concrete frame or structure. The following specification is recommended by the American Society of Civil Engineers:

Portland Cement

Definition: This term is applied to the finely pulverized product resulting from the calcination to incipient fusion of an intimate mixture of properly proportioned argillaceous and calcareous materials, and to which no addition greater than 3% has been made subsequent to calcination.

Specific Gravity

The specific gravity of cement shall not be less than 3.10. Should the test of cement as received fall below this requirement, a second test may be made upon a sample ignited at a low red heat. The loss in weight of the ignited cement shall not exceed 4%.

Fineness

It shall leave by weight a residue of not more than 8% on the No. 100, and not more than 25% on the No. 200 sieve.

Time of Setting

It shall not develop initial set in less than thirty minutes; and must develop hard set in not less than one hour, nor more than ten hours.

Tensile Strength

The minimum requirements for tensile strength for briquettes one square inch in cross section shall be as follows and the cement shall show no retrogression in strength within the periods specified.

Neat Cement

Age	Strength
24 hours in moist air	175 lbs.
7 days (1 day in moist air, 6 days in water)	500 lbs.
28 days (1 day in moist air, 27 days in water)	600 lbs.

One Part Cement, Three Parts Standard Ottawa Sand

7 days (1 day in moist air, 6 days in water)	200 lbs.
28 days (1 day in moist air, 27 days in water)	275 lbs.

Constancy of Volume

Pats of neat cement about three inches in diameter, one-half inch thick at the center, and tapering to a thin edge, shall be kept in moist air for a period of twenty-four hours.

(*a*) A pat is then kept in air at normal temperature and observed at intervals for at least 28 days.

(*b*) Another pat is kept in water maintained as near 70° F. as practicable, and observed at intervals for at least 28 days.

(*c*) A third pat is exposed in any convenient way in an atmosphere of steam, above boiling water, in a loosely closed vessel for five hours.

These pats, to satisfactorily pass the requirements, shall remain firm and hard and show no signs of distortion, checking, cracking or disintegrating.

Sulphuric Acid and Magnesia

The cement shall not contain more than 1.75% of anhydrous sulphuric acid (SO_3), nor more than 4% of magnesia (MgO).

4. Quick Tests

The preceding specifications and methods of investigation presuppose the conveniences of a testing laboratory to be at hand. The constructor, when he comes upon a job, is frequently without such conveniences. He is frequently compelled to decide whether the cement is suitable or not by employing such rough and ready means only as are at hand, and a few words as to such practical methods of investigation as must be used are in order for his benefit.

Fineness

The constructor can readily determine whether the grinding is reasonably coarse or not by feeling of a sample between the thumb and finger without recourse to screens or sieves.

The effect of fine grinding upon the cement is to render samples of mortar made of sand and cement stronger. In other words, it gives the cement a greater sand carrying power; it renders it quicker setting; a stronger concrete is obtained, or a larger proportion of sand can be used with finely than with coarsely ground cement with the same resulting strength.

In making briquettes of neat cement, however, the coarsely ground cement may show higher results, but what the constructor is interested in is the result obtained with the mortar paste of sand and cement in the usual proportions.

Accelerated Test

The object of this test is to bring out and make evident those qualities which tend to destroy the strength and durability of a cement. As it is highly essential to determine such qualities at once, tests of this character are for the most part made in a very short time, and are known, therefore, as accelerated tests. Failure is revealed by cracking, checking, swelling or disintegration, or all of these phenomena. A cement which remains perfectly sound is said to be of *Constant Volume.*

Failure to meet the requirements of the accelerated tests in shipments direct from mill need not be sufficient ground for rejection. The cement may be held for twenty-eight days and a re-test made at the end of that period. But failure to meet the requirements at this time should be considered sufficient cause for rejection.

The accelerated test is a rough and ready means for determining without elaborate equipment whether cement is fit to use. Cement known to have been stored by a dealer for some time should be promptly rejected if it fails in this test.

If a Portland Cement passes the accelerated test it may be used immediately with reasonable certainty as to its ultimate soundness.

Method of Testing

The method of making the accelerated test, is as follows: On a piece of glass about four inches square, take a sample of the cement and mix it to a consistency such that the cement can be readily kneaded without crumbling and at the same time not so soft as to run or lose its shape when pressed into a smooth pat with a thin edge. Place the pat so formed under a moist cloth for a period of twenty four hours in a temperature from sixty to seventy five degrees and then expose it to an atmosphere of steam. Or, if preferred, the specimen after curing as above for twenty four hours may be placed in cold water, which is raised to and maintained at the boiling point for several hours. Three to four hours is the usual period. Under this test the pat should harden without cracking or swelling.

Causes of Unsoundness

Cracking, crumbling, or disintegration of work in Portland Cement concrete properly mixed and laid may be caused by an excess of lime; by under burning or by an excess of magnesia in a thoroly burned cement, producing gradual expansion which will disintegrate the mortar or concrete even after several years.

Care of Cement

The inspector should see that the cement is properly housed when delivered to the job and protected from the elements so that it will not be damaged by moisture. Dampness from insufficient protection will render the cement lumpy and while it may not destroy its setting properties it will greatly reduce its sand carrying power and efficiency or may even render it entirely worthless.

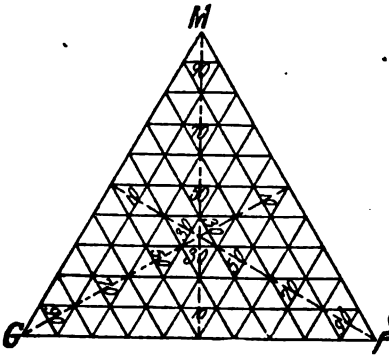
5. Specification for Aggregate

Sand: Sand used should be clean and coarse, or a mixture of coarse and fine grains with coarse grains predominating, which should be free from clay, loam, mica and other impurities.

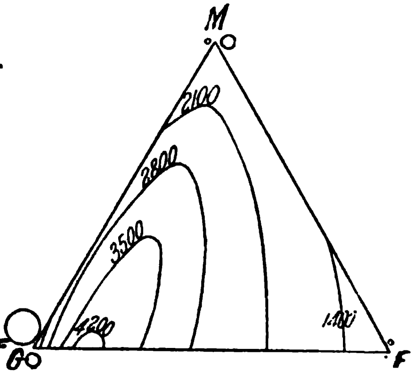
Testing Sand: In order to determine the amount of clay, dirt or other impurities, a simple, practical test is to take an ordinary quart glass preserve jar, put in a pint of sand, fill with water and put on the cap. Shake thoroughly and allow it to settle. The result will be that the coarser grains will go to the bottom in the order of their size, and the silt and light impurities will settle in a layer at the top, giving the observer a means of gaging the amount of the impurities accurately and judging of the character of the sand and the proportion of coarse, medium and fine grains in its make-up. From three and one-half to four percent of clay in the form of finely divided silt will do no harm in a bank sand or gravel for reinforced concrete work. Even higher percentages than this have been claimed to increase the strength of the concrete under test, though where it is exposed to the elements and the action of frost a percentage even as high as this seems to be quite detrimental. However, in building work, which is usually under cover, it does no harm whatever.

The effect of the size of the grains of sand has been investigated by Feret. The accompanying figure from Johnson's Materials of Construction, shows results obtained by Feret on a 1 : 3 mortar after hardening one year in fresh water. The sand used consisted of various proportions of fine grains up to .5 mm, medium .5 to 2mm, and coarse 2 to 5mm, and in the diagram the strength of the mortar is recorded in the triangle at such distances from the base line as represent the proportions of each size of sand used, the line of equal strength being wherever drawn in the diagram. Thus the strength of the mortar in which only fine sand was used was only 1400 pounds per square inch. The maximum strength of 3500 pounds per square inch was obtained from a mixture containing 85 percent of coarse sand and 15 percent of fine with a very little sand of medium size.

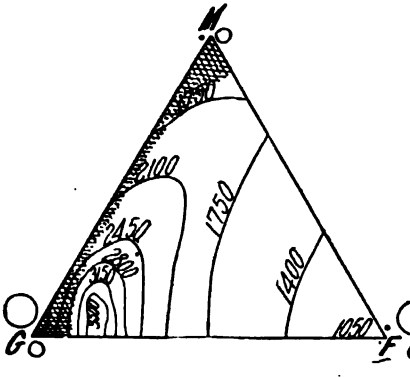
Any point of an entire contour line represents a sand made up of the different sizes G, M, and F in proportions corresponding to its perpendicular distance from the sides opposite each apex but having the same strength as every other point on the same line. This diagram shows that a considerable variation in the proportion of coarse and fine grains will make a mortar of the same strength, but that, in general, the strength of a mortar with fine sand of uniform size of grains is about one half or less than one half that of a mortar made with the same proportion of sand with grains ranging from coarse to fine, and that in general the strongest mortar is secured with a coarse sand with grains ranging from coarse to medium.



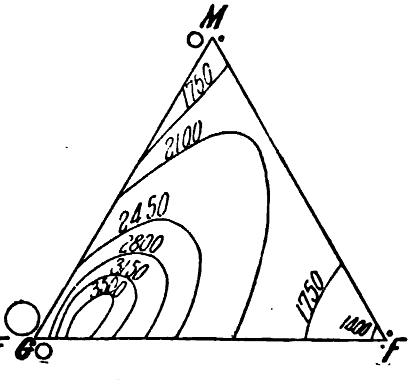
—Showing the Method of Representing Proportionate Mixtures of Three Ingredients. G=coarse sand, 0.2 in. to 0.08 in. in diameter. M=medium sand, 0.08 in. to 0.02 in. in diameter F=fine sand less than 0.02 in. in diameter.



—Compressive Resistance of Portland-cement Mortars, in pounds per square inch, after nine months in air and then three months in sea-water. Mortar 1 C. : 3 S. in all cases, but the composition of the sand varying according to position in the triangle.



—Compressive Resistance of Portland-cement Mortars, 1 C. : 3 S. in pounds per square inch, *after one year in sea-water.* Shaded part indicates mixtures which were partially disintegrated.



—Compressive Resistance of Portland-cement Mortars, 1 C. : 3 S., in pounds per square inch, *after one year in fresh water.*

The effect of an excess of clay, such for instance as the dust from soft magnesium limestone will sometimes greatly retard the hardening of cement, the writer having seen instances where concrete at the age of a month had not attained twenty five percent of its normal strength and where the ultimate strength was reduced perhaps not more than twenty five or thirty percent by the use of this improper mixture.

In many specifications, clean sharp sand is called for in spite of the fact that in many parts of the country sharp sand is not obtainable. Sand with rounded grains such as lake or beach sand is perfectly satisfactory, there being little difference in strength between the mortar made with sand of angular or sharp grains and that with rounded grains. The idea that a sharp, angular aggregate is necessary for strong concrete is the basis for the objection made by some to lake or bank gravel as a coarse aggregate, while as a matter of fact better results and stronger concrete is generally secured with the round pebble than with angular stone, provided the specimen tested is not less than six months old.

Gravel where used, should be composed of clean, hard pebbles and sand free from clay and other foreign matter, such as rotten stone, hardened lumps of clay and the like. A sample having the coarser materials screened out may be tested for impurities in the same manner as was given for the sand.

Broken Stone: Broken stone used should consist of sound crushed stone, such as trap rock, limestone, granite, hard sandstone or conglomerate. If the texture is crystalline and there are no portions of rotten stone or hardened clay such as is sometimes found in oölitic limestone and shale, the crusher run may be used, if a part of the sand content which would otherwise be used in the mix be left out equivalent to the fine particles of crushed stone in the crusher run.

If, however, the stone can be readily reduced under the hammer to a fine, impalpable powder as is the case with some shales and with the oölitic type of limestone referred to, the dust should be entirely removed.

It is better, where possible to use only that stone which is found durable when exposed to the action of the elements and frosts, and the harder the stone the stronger the concrete that may be made when using it as the aggregate.

6. Proportions of Materials

In concrete building construction the proportions which experience indicates most economical in concrete for slab and beam construction, columns and column footings except for the case where the loads to be carried are unusually great, is one part of cement and two parts sand with four parts broken stone or gravel, this being indicated by the expression 1 : 2: 4. These proportions are customarily taken by measure, each bag of cement being estimated as equal to one cubic foot in volume, thus the proportions of 1: 2: 4 mean one sack of cement (94 pounds) two cubic feet of sand and four cubic feet of crushed stone. Too frequently the inexperienced builder interprets a 1 : 2 : 4 concrete to mean a 1 : 6 aggregate or six parts of gravel which may be two thirds sand and one third coarse aggregate. The proportions of 1 : 2 refer to the mortar and mean the ratio of the cement to the volume of sand which runs from about one eighth inch down in size, while coarser material than this may be considered as coarse aggregate or the stone content.

With lean mixtures of concrete in floor slabs having a thickness of less than $L/12$, repeated loadings cause the floor to take small permanent sets continuing through a period of years. Slabs of 20′ span from 8 to 8½″ in thickness of as lean a mixture as one cement to six bank run gravel, containing 3½ to 4 parts sand, have sagged out of shape from 1″ to 1¾″ and while they are in continuous use and safe against collapse such deformation is so unsatisfactory that the lean mixture which produced it should be avoided.

The size of the stone in reinforced concrete work in ordinary building construction should range from one inch down, observing directions for screening as detailed under the specification for broken stone.

The first requirement in proportioning the aggregate for reinforced concrete work is to see that there is an excess of fine material over and above that required to fill the voids in the coarse component of the aggregate. The volume of voids in coarse aggregate is greater with a uniform size of stone than where the sizes of coarse aggregate vary from coarse to fine, and for that reason the crusher run of stone is preferable where the stone is granite, trap or hard crystalline stone. In heavy mass work, however, a larger proportion of coarse aggregate with the size of stone varying from three to four inches in diameter down can be advantageously used, but this is unsuited for reinforced concrete work in the ordinary building line. It is only suitable for bridge piers, and heavy mass work.

Adoption of an arbitrary proportion of cement to sand in the mortar with the idea of securing a standard degree of strength requires the exercise of judgment in the selection of the sand. The grains of sand should not be too uniform in size as indicated by the diagrams on the preceding page. A mixture of different grades, if available, obviates this difficulty. Where the sand available is too fine additional cement must be used to make good this deficiency.

The determination of the permissible proportion of coarse to fine aggregate by finding the volume of water required to fill the voids in a unit volume of stone and the adoption of that volume of sand equal to the volume of water so determined has been suggested. On this theory if the stone or gravel contains 40 per cent of voids the sand should be 40 percent of the volume of the stone for the proportion of one cement, two sand and five stone. Loose measurement is generally adopted because it corresponds more nearly to the final volume of the material in the concrete in which more mortar is required than would just fill the voids in the stone when settled in a measure by shaking. Because the mortar must surround each particle of stone and encase it to form sound concrete, the particles of stone are separated by this envelope so that they occupy a greater volume. Thus approximately ten percent of sand in excess of the volume required to fill the voids in the stone is necessary to permit the mixture to flow into place and form a solid mass. Where the voids measured are forty percent of the volume of the stone the mixture, instead of being a 1 : 2 : 5, should be for the above reason in the proportion of 1 : 2 : 4½ by volume.

The method of porportion by voids, determined as above, while apparently logical, is of little practical value for the reasons explained. Careful observation of the manner in which the concrete works as it is poured or flows into position will enable the engineer to judge more accuately whether arbitrarily assumed proportions with the given materials may be modified to advantage.

7. Analysis of Strength of Concrete

Concrete may be defined as an artificial conglomerate stone in which the coarse aggregate or space filler (generally a hard natural stone, furnace slag or pebble) is held together by a cement mortar matrix. Having selected a given coarse aggregate, the strength of the concrete depends on the strength of the mortar matrix, in other words, on the ratio of cement to sand in the mortar for all samples of the same age, formed under the same conditions.

The Strength of the Concrete depends then:

First, on the grade of sand and the proportion of the cement to the sand in the mortar.

Second, upon the hardness and the character of the coarse aggregate.

Third, on manipulation and the conditions under which the concrete is cured or hardened.

Fourth, on the age of the specimen.

Mortar made with a very fine sand is only about half as strong as that made with coarse and medium grains and for that reason the specification regarding the character of the sand should be given careful attention.

As shown by Feret, quite a variation in the proportion of medium, coarse and fine grains of sand will give nearly the same strength so that the average clean coarse bank sand will generally fill the requirements for a good concrete mortar.

The richer the mortar the stronger the concrete. As noted above, we recommend a one-to-two mortar for reinforced concrete, and where high working stresses are to be used in reinforced concrete columns we would recommend a mortar in the concrete as rich as one cement to one and one half sand, obtaining thereby an increase of twenty five percent in the permissible working stress.

Coarse Aggregate

The effect of the strength of the coarse aggregate upon the strength of the concrete, in tests of concrete made with shale rock crushed to one and one half inches or under, at Duluth, show the shale concrete about sixty-five to seventy percent as strong as trap rock concrete and the trap rock concrete from ninety to ninety-five percent as strong as that made with lake gravel for the coarse aggregate. These tests were made on concrete about four months old.

Manipulation and Conditions of Curing

While the quality of the cement, sand and aggregate have more or less influence on the resulting concrete, with any good brand of first class Portland Cement, clean coarse sand and hard crushed stone, substantially the same results will be secured under identical conditions of mixing and curing. The latter conditions have a most decided influence on the strength of the concrete, viz., whether sufficient water has been used to permit and promote perfect crystallization of the cement, whether an excess amount of water has been used and the fine and coarse materials have been allowed to separate or become segregated, whether the concrete has been thoroly mixed, and whether the conditions for curing were favorable, such as keeping the concrete damp and preventing it from drying out too rapidly or whether it was hardened under unfavorable circumstances in frosty weather. On this account it is difficult to harmonize the large number of isolated tests that have been made by independent investigators under widely varying conditions.

In building work, however, it is a fortunate fact that except in cold weather where the work requires special treatment the general conditions for hardening are most favorable. After one floor has been poured the next is erected thereon within a week or so and the excess water dropping from the upper floors keeps the concrete in the lower properly wet, rendering the conditions more favorable for hardening and curing than those of the ordinary laboratory test.

8. Hardening of Portland Cement

The hardening of Portland Cement is a chemical process which within certain limits is accelerated by heat and retarded by cold. This is an important consideration for the builder to keep in mind, since when the temperature of the water used in the mix and the aggregate is at or approximately near the freezing point, the cement lies dormant and no fixed rule can be given of a set number of days during which time it is necessary for the concrete to lie in place on the forms before it will attain a certain given degree of strength. In hardening, as in nearly all chemical reactions, heat is generated by the setting of the cement. This heat is radiated away very rapidly where the mass is small or thickness of the part of the concrete work is inconsiderable, while where the mass is large, as in the case of heavy walls, piers, and the like, the heat generated by the setting of the cement is not lost rapidly by radiation and the work tends to cure much more rapidly in heavy work in cold weather than in the case of the thin slabs usually used in the floors of buildings. Special directions for the treatment of concrete at various temperatures will be given under a special section dealing with construction work.

Increase of Strength of Concrete with Age

The following table shows compressive strength of concrete as determined by test made at the Watertown Arsenal in 1899. 1 : 2 : 4 mixture.

Brand of cement	7 days	1 month	3 months	6 months
Atlas	1,387	2,428	2,966	3,953
Alpha	904	2,420	3,123	4,411
Germania	2,219	2,642	3,082	3,643
Alsen	1,592	2,269	2,608	3,612
Average	1,525	2,240	2,944	3,904

The above gives a fair idea of the increase in strength of concrete with age under normal temperature above 60° F.

After a period of six months the concrete in ordinary building is found to increase slowly in strength and considerably in hardness and rigidity. Thus it appears that the stiffness of a long span slab will increase about twenty percent between two months and twelve to fifteen months, and the strength perhaps in a lesser ratio, on the assumption that the compression element only in the combination is hardening and increasing in strength.

9. Adhesion, Bond Shear

The strength of the composite structure of concrete and metal depends on the resistance which the embedded rods offer to longitudinal motion or sliding within the concrete matrix. This resistance is termed adhesion or bond. The resistance to sliding is obviously a shearing resistance and will be termed *bond shear* in the ensuing discussion. Concrete setting in the air shrinks and grips the reinforced members with a vice-like grip. The richer the mixture the greater this shrinkage stress and the better the bond. In concrete setting in water this shrinkage is lacking and in this case deformed reinforcement or mechanical bond is desirable.

The bond between the concrete and steel has a maximum value with a plastic mix of concrete such that the mortar will flow slowly and thoroly surround the metal. It is greatly reduced with a stiff mixture requiring tamping and at the other extreme also is less with too sloppy a mixture of concrete.

With plain round rods embedded 12 inches the bond value may reach, under favorable conditions, a maximum of 600 pounds per square inch with concrete of a 1 : 2 mortar, six months old, but with dry tamped concrete the bond value or adhesion as it is sometimes called may run as low as 200 pounds per square inch of the surface of the bars.

A round bar will give a higher bond value than a flat or rectangular shape, while a polished or cold rolled shaft as it comes from the mill will give a value hardly more than a fourth as great as that. Slight rusting of the surface improves the adhesion or bond since the rust combines chemically with the cement and seems to increase the shrinkage grip. Further, slight rusting tends to remove the black mill scale making the adhesion uniform along the surface of the metal.

Paint, oil or grease, greatly reduces the adhesion. With properly arranged reinforcement the designer rarely has occasion to figure upon the bond value between the two materials, since it is amply provided for where due precaution has been taken to render the design safe to execute by properly tying the materials together by an ample lap of the metal over the supports and the use of *sufficient cement.*

Occasionally the designer may be forced to use short stock lengths in beams or slabs, and under such conditions a working stress not exceeding forty pounds per square inch is permissible providing the bars are also hooked at the ends. Even with this additional precaution care must be exercised in keeping the work supported much longer than would be necessary with preferable lengths of rods. The reason for this precaution and the low value recommended is that the bond strength between concrete and steel varies greatly with the age of the concrete and like the shearing resistance it is very low with partly cured concrete but increases with the age and hardness of the work, tho less rapidly than the resistance in compression.

The constructor should keep clearly in mind the important conditions which insure satisfactory bond between the metal and concrete, namely: A mixture of *proper consistency containing sufficient cement* and the preferable use of bars round in section, which are most readily surrounded by the plastic concrete in flowing.

It is interesting to note in the following table of tests of bond that the average adhesion of the ½″ round rods was twenty percent more than that of the square rods with the 1 : 3 mortar; that the adhesion to the steel with the broken stone concrete was greater than with the 1 : 3 mortar or even the neat cement test; and that the adhesion to the quarter inch by one inch averaged only six tenths that of the half inch rounds.

TABLE I

SHOWING ADHESION OF VARIOUS SHAPED RODS TO CONCRETE OF CONSTANT COMPOSITION

1 Part Cement to 3 Parts Sand
*9–10–11–12, 1 Cement, 2 Sand, 4 Broken Stone
(Even numbers 40 days—Odd Numbers 80 Days)

No.	Section of Steel	Length Embedment ins.	Perimeter, ins.	Load in lbs. at Failure	Area of contact sq. ins.	Load in lbs. per sq. ins.	Average per sq. in.
1	$\frac{1}{2}''$ square....	$6\frac{1}{8}$	.....	.....	12.25	...	...
2		$6\frac{1}{2}$	2.0	5,700	13.00	438	432
3		6	.....	5,200	12.00	433	...
4		$6\frac{1}{4}$	.....	5,300	12.50	424	...
5	$\frac{1}{2}''$ round....	6	.....	4,600	9.43	488	...
6		6	1.571	5,000	9.43	530	512
7		6	.....	4,600	9.43	488	...
8		$5\frac{7}{8}$	.....	5,000	9.23	542	...
9	$\frac{1}{4}''$ x 1''....	$6\frac{1}{4}$	.....	4,900	15.63	313	...
10		$6\frac{1}{4}$	2.5	4,400	15.63	282	293
11		$6\frac{1}{8}$	.....	4,800	15.20	314	...
12		$6\frac{1}{4}$	.....	4,400	15.63	282	...
5	1'' square...	$10\frac{1}{4}$	.....	17,400	41.0	424	...
6		$10\frac{1}{8}$	.....	15,800	40.5	390	411
7		$10\frac{1}{8}$	.....	17,000	40.5	420	...
8		$10\frac{1}{4}$	.....	16,800	41.0	410	...
* 9		$10\frac{1}{8}$	.....	21,200	40.5	523	...
*10		$10\frac{1}{4}$	.....	24,600	41.0	600	587
*11		$10\frac{1}{8}$	.....	24,200	40.5	598	...
*12		$10\frac{3}{8}$	.....	26,000	41.5	627	...

Test by Emerson, Eng. News, 1904, p. 222.

The adhesive resistance varies somewhat with the depth to which the rod is embedded; tests by Feret with the rod embedded $2\frac{3}{4}''$ give values approximately half as great as where the length embedded is from ten to twelve inches.

It is greater with a rough than with a smooth surface.

It increases with the proportion of cement up to a certain limit.

It is a maximum with a plastic mix and a minimum with a dry mix and tamped concrete.

It increases with the age of the concrete.

Considère finds that for concrete exposed to air the amount of water used in mixing has a great influence, too dry concrete adhering badly. An excess of water giving the concrete the necessary fluidity for filling up the voids around the reinforcement produced the best results. He considered, however, that this advantage of wet concrete was counterbalanced by a notable dimunition of tensile and compressive resistance. This would be true were it not a fact that the excess water in casting reinforced concrete work in building construction is readily disposed of by absorption of the forms and leakage through them.

The low values found by some investigators for adhesion or bond seem to be readily accounted for by the prevalent French custom of tamping dry or stiff mixtures of concrete rather than of pouring it of a wet or plastic consistency, as is done by the American constructor today. The early idea was that good concrete could only be made by a dry mix and tamping. Combined with this dry mix the deformed bar was unquestionably an improvement, but as the use of the dry mix has long been abandoned the main advantage in the use of the deformed bar has largely disappeared with its abandonment.

That the bond between the concrete and steel is really a shrinkage grip may be easily proved by the simple experiment of molding some concrete and placing a piece of round steel on top, lightly pressing it into the concrete without immersing it more than one third the diameter. When the concrete has cured it will be found that there is very little difficulty in removing the steel. If the piece, however, is pressed into the concrete to a considerable depth and the concrete in its plastic condition allowed to flow around the bar it will be very difficult indeed to remove and it will be found that this is caused by the shrinkage of the concrete around the bar in hardening. That this bond between the concrete and the rod is not due to direct adhesion may be further proved by splitting the concrete about the bar or sawing it down to the side of the bar on each side when it will be found that the bar is readily removed from the concrete.

Adhesion or bond in reinforced concrete is unaffected by temperature changes, since the coefficient of expansion of concrete and steel is substantially the same, to wit. .0000065. The composite mass, however, is subjected to temperature stresses just as any building material would be under like conditions, and because of the small tensile strength of concrete, reinforcement to resist these stresses is commonly required as will be discussed more at length later.

10. Variation in the Strength of Concrete with Variation of Temperature and Moisture.

This question is of the greatest importance to the constructor in putting up work. The concrete, partly cured, may apparently be stiff and rigid when the forms are removed in cold or freezing weather. Then with a sudden change in the temperature, such as may readily be brought about by putting a heating plant into the building, the concrete will sweat and soften and get out of shape. Again, when concrete which has had as long as two or three months in which to cure during the fall season, is exposed all winter, soaked with water and the water frozen, and in the spring a heating plant is put into the building and the slab thawed out, its strength is temporarily greatly reduced, and if the slab is carrying the weight of other stories which are shored from it, permanent deflection and serious trouble may result.

The older and more thoroly cured the concrete, the more rigid it is and the less the variation in strength resulting from the conditions above noted. Concrete which is thoroly soaked with water is less rigid in compression than concrete which is thoroly dried out. This change in strength is due to the fact that the concrete expands with moisture and shrinks or contracts as it dries out, this action being greater with concrete during the hardening stages.

Hence, the constructor should use care and see that his work is not over-loaded, particularly at the time when he is firing up the heating plant in a building in which the floors, though they have had some time to cure in the fall, have been thoroly soaked and frozen. Undue confidence in the strength of partly cured and frozen concrete arises from observing that when the forms are first removed there is no deflection and the concrete stands up apparently of ample strength and rigidity under the superposed load of the centering of one or more stories above. A sudden change in temperature, causing the moisture in the slab to thaw and expand in the concrete will so weaken the slab that a large permanent set will result. The cautious builder will keep the floor well shored up until he is sure that it is thoroly dried out by heat so that there is no chance for the work to get out of shape as above explained.

The extent to which the strength of a slab may be reduced even after it has been once fairly well cured, but subsequently exposed to the weather, soaked and frozen is illustrated by a case where it was so softened in thawing out as to deflect eight times as much as it did under identically the same load after drying out and exposure to heat for three weeks, so that the importance of the precaution above outlined should be apparent. In this case the slab was cast in the latter part of August, was fairly well cured but exposed to the weather, flooded by rain and frozen up during the winter. The heating plant was placed under the floor sometime in March and a light load then placed on the slab which had a span of about 22 feet and was 7 inches thick, well reinforced. A deflection resulted of approximately 2 inches. The slab returned to its original shape after removal of the load, and when thoroly dried out the deflection under the same load was hardly 3/16 inches.

Deflections are found to be increased where the slab is wet, and the strength is apparently somewhat diminished.

11. Machine Mixing.

Concrete for a concrete steel building should be machine mixed, preferably in a batch mixer. Some of the continuous mixers do good work where bank gravel is used as the aggregate and fail where crushed stone is used. A batch mixer such as the Smith, Cube, Polygonal or Ransome, is to be preferred because it may be charged with cement, sand and stone by measure and the exact amount of water added. The water content in the mix is best supplied for a large piece of work by a tank which will contain the amount of water needed for a batch arranged with the usual float trap valve so that all the operator needs to do is to pull the string and the tank of water is discharged at once in the mixer. This insures a mixture of uniform consistency and effects a material saving of time.

Consistency of Concrete

For building construction and reinforced concrete work generally it is necessary that the concrete shall be mixed so that it will flow slowly and thoroly surround the reinforcement but it should be no more plastic than is required to attain this result. If mixed too dry and tamping is depended upon, voids will be left around the steel and the face of the concrete when the forms are removed will be found rough and full of pockets and the work will present an appearance of weakness which it very likely does not possess.

12. Pouring Concrete.

In pouring concrete, the lowest portions of the forms should be filled first; thus permitting the least possible flow of the concrete to reach it final position in the work. In buildings the columns should be filled first, then the beams and finally the slabs, the operation being continuous as far as practicable. If an attempt is made to reverse this program and fill the beam before the column is filled the concrete will flow in an inclined direction to the column and as each batch is deposited the more liquid portions washing over the inclined surface carry the light laitance and fine sand down into the column, and an inferior concrete and one of little strength will be found at the bottom on removal of the column forms.

In pouring columns, especially where closely spaced spiral hooping is used, concrete should be poured over the center. If an attempt is made to fill the column from the side the space between the form and hooping is filled up to a considerable height in advance of the central core, the hooping acting as a screen prevents the coarse aggregate from flowing to the lower central area with the result that the mortar flows to the core and seals to some extent the voids toward the hooping and the next batch cannot flow to fill in the voids thus left in nearly clean coarse aggregate between the hooping and the column form. This leaves rough unsightly work when the forms are removed and while the core may be sound the fireprrofing is inferior in character or worthless, and the work presents an appearance of weakness which it does not possess.

Where to Make Joints in the Work and How to Do It

Splicing of concrete in beams and slabs should preferably be made in the center and should be vertical. The reason for this is that where the concrete is allowed to flow out on an inclined plane in the beam the inert material known as laitance comes to the surface, preventing a good bond when the new concrete is added. In fact instances have not infrequently been observed where wedge shaped pieces of concrete three feet long and running from two inches in thickness to one quarter inch at the end dropped away from the beam owing to this manner of placing, the bond being insufficient to carry the weight of the piece. The remedy is to break up the surface of the old concrete and grout it with a neat cement before proceeding to cast the new work.

In very hot weather the crushed stone may readily get so dry and hot that it absorbs the water from the mix and causes the cement to set too rapidly. When this is the case an open crack will appear at a joint. The remedy is to first cool the stone by thoro wetting with the hose.

13. Reinforcing Steel

Steel for reinforcement should be tough, homogeneous metal, preferably of structural or intermediate grade, where bending is required, or of harder grade for slab rods when bending is unnecessary.

The following are the specifications adopted by the Association of American Steel Manufacturers, 1910, governing the chemical and physical properties of concrete reinforcing bars:

Standard Specifications for Concrete Reinforcement Bars

1. *Manufacture.* Steel may be made by either the open-hearth or Bessemer process. Bars shall be rolled from billets.

2. *Chemical and Physical Properties.* The chemical and physical properties shall conform to the following limits:

3. *Chemical Determinations.* In order to determine if the material conforms to the chemical limitations prescribed in paragraph 2, herein, analysis shall be made by the manufacturer from a test ingot taken at the time of the pouring of each melt or blow of steel, and a correct copy of such analysis shall be furnished to the engineer or his inspector.

4. *Yield Point.* For the purposes of these specifications, the yield point shall be determined by careful observation of the drop of the beam of the testing machine, or by other equally accurate method.

5. *Forms of Specimens.* (*a*) Tensile and bending test specimens may be cut from bars as rolled, but tensile and bending test specimens of deformed bars may be planed or turned for a length of at least 9 inches if deemed necessary by the manufacturer in order to obtain uniform cross-section.

Properties Considered	Structural Steel Grade		Hard Grade		Cold-Twisted Bars
	Plain Bars	Deformed Bars	Plain Bars	Deformed Bars	
Phosphorus, maximum,					
Bessemer.........	.10	.10	.10	.10	.10
Open-hearth......	.06	.06	.06	.06	.06
Ultimate tensile strength, pounds per sq. in.........	55–70,000	55–70,000	80,000 min.	80,000 min.	Recorded only
Yield point, minimum pounds per sq. in .	33,000	33,000	50,000	50,000	55,000
Elongation, per cent in 8″, minimum...	$\frac{1{,}400{,}000}{\text{T. S.}}$	$\frac{1{,}250{,}000}{\text{T. S.}}$	$\frac{1{,}200{,}000}{\text{T. S.}}$	$\frac{1{,}000{,}000}{\text{T. S.}}$	5%
Cold bend without fracture:					
Bars under ¾″ in diameter or thickness............	180°d.=1t.	180°d.=1t.	180°d.=3t.	180°d.=4t.	180°d.=2t.
Bars ¾″ in diameter or thickness and over............	180°d.=1t.	180°d.=2t.	90°d.=3t.	90°d.=4t.	180°d.=3t.

The hard grade will be used only when specified.

(*b*) Tensile and bending test specimens of cold-twisted bars shall be cut from the bars after twisting, and shall be tested in full size without further treatment, unless otherwise specified as in (c), in which case the conditions therein stipulated shall govern.

(*c*) If it is desired that the testing and acceptance for cold-twisted bars be made upon the hot rolled bars before being twisted, the hot rolled bars shall meet the requirements of the structural steel grade for plain bars shown in this specification.

6. *Number of Tests.* At least one tensile and one bending test shall be made from each melt of open-hearth steel rolled, and from each blow or lot of ten tons of Bessemer steel rolled. In case bars differing ⅜ inch and more in diameter or thickness are rolled from one melt or blow, a test shall be made from the thickest and thinnest material rolled. Should either of these test specimens develop flaws, or should the tensile test specimen break outside of the middle third of its gauged length, it may be discarded and another test specimen substituted therefor. In case a tensile test specimen does not meet the specifications, an additional test may be made.

(*d*) The bending test may be made by pressure or by light blows.

7. *Modifications in Elongation for Thin and Thick Material.* For bars less than 7/16 inch and more than ¾ inch nominal diameter or thickness, the following modifications shall be made in the requirements for elongation:

(*e*) For each increase of ⅛ inch in diameter or thickness above ¾ inch, a deduction of 1 shall be made from the specified percentage of elongation.

(*f*) For each decrease of 1/16 inch in diameter or thickness below 7/16 inch, a deduction of 1 shall be made from the specified percentage of elongation.

(*g*) The above modifications in elongation shall not apply to cold-twisted bars.

8. *Number of Tests.* Cold-twisted bars shall be twisted cold with one complete twist in a length equal to not more than 12 times the thickness of the bar.

9. *Finish.* Material must be free from injurious seams, flaws, or cracks, and have a workmanlike finish.

10. *Variation in Weight.* Bars for reinforcement are subject to rejection if the actual weight of any lot varies more than 5 percent over or under the theoretical weight of that lot.

20. Quality of Steel

Unfortunately there is an idea prevalent that almost any grade of metal is good enough for reinforcement. Where the contractor or engineer is responsible for the test strength and permanence of the work he needs to see that the steel is of suitable quality.

The product of what is called a fagot mill is generally very undesirable. The trade term, sometimes applied to this product is "Bushel Steel." A fagot is formed using muck bar iron flats for bottom and sides and filling in with miscellaneous scrap steel, iron, etc., heating up and rolling into billets and bars. This grade of metal has an ultimate strength of about 45,000 pounds per square inch and a commercial yield point of 25,000 pounds. It will bend easily but when nicked and broken will show a dull fracture of a ragged and coarse texture in strong contrast with the bright, fine crystalline or silky texture of a good grade of steel.

Rerolled rail steel is sold to a considerable extent for reinforcing metal. Because of the rigid specification insisted upon by the railroads for rails purchased in the last seven or eight years, accepted rail steel is fairly uniform in quality. The effect of rerolling it, however, may be and sometimes is very detrimental to the toughness and dependability of the material. Rerolled steel differs from new billet stock of the same carbon as the result or effect of the heat treatment preceding rolling. A rail differs from a rectangular billet as follows:

The rectangular billet in the furnace heats with greater uniformity than an irregular section such as a rail presenting difference in thickness of the head, flange, and stem. For rerolling the rails are heated in the furnace to a pasty white heat, then the flange and head are sheared from the stem by splitting rolls and each section is in turn rolled into the bars or shapes of different sizes. Where the heat has been too intense, exterior surfaces of the portions of the rails may be overheated while the inner portions of the head are not heated uniformly, so that the bars rolled from the head are brittle, crack in bending and fail to show under test the elongation of the original rail steel.

Trouble from this source seems to occur where oil furnaces are used for heating in which the control of the heat frequently is not as satisfactory as where the fuel used is coal. On the other hand it appears that the product of the mill where the heat treatment is scientifically handled may not be greatly inferior to high carbon standard billet stock under hard grade specifications.

The rail head under traffic is battered and the metal flows to the outer side of the head, leaving splinters or slivers and these must be knocked off with a sledge before heating the section of rails for rerolling, otherwise the finished surface will be imperfect. presenting splinters and flaws. Where attention has been given to this detail, the surface of the bars rerolled from old rails is as satisfactory as that secured from rerolling new rails in the same mill.

Twisting a square bar to increase the bond is practical only with a soft structural grade of steel. Cold twisting renders the metal more brittle, raises the yield point, reduces its toughness and elongation, and is for that reason objectionable. Where the mechanical bond of a twisted bar is demanded, hot twisting is preferable for the above reasons.

Hard grade steel has a decided advantage for slab rods of small diameter, since the harder the metal the less liable they are to kink in the handling and shipping. Specifications for small bars such as 5/16 to ½ inch, should require that these bars be shipped in bundles of a dozen to fifteen rods per bundle well wired together, so that they will be less liable to be bent in shipment and can be more readily handled on the work.

CHAPTER II

GENERAL TYPES OF CONCRETE FLOOR CONSTRUCTION

1. Classification—The history of the development of structural work shows that the engineer has been largely influenced in his first efforts to design any new type by the forms of construction to which he has been previously accustomed. For example, when wrought iron began to be used in place of timber for railroad trestles the longitudinal bracing was identical with that used in timber construction; indeed it was at first gravely questioned whether these braces ought not to be made of timber for fear of the unknown dangers that might arise from the unequal expansion of these braces of iron and the foundation on which the trestle was supported, and

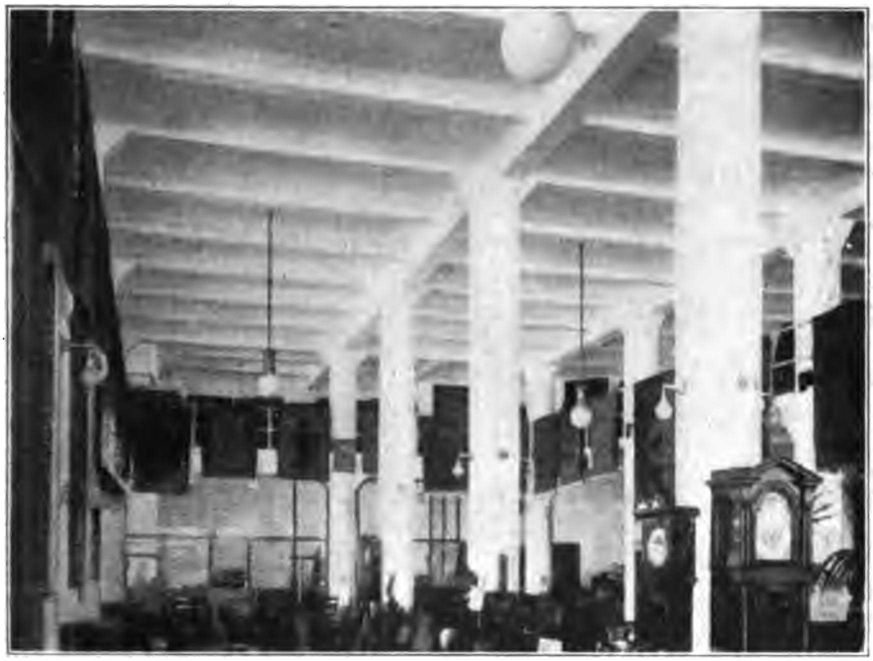


Fig. 1. Type I.

today not a few of our concrete theorists are deeply concerned regarding equally insignificant questions.

The common types of concrete steel floor construction may be classified as follows:

I. The earliest type of timber construction has been followed or imitated closely in some of the pioneer structures in concrete steel and also in not a few of our buildings even today. This type may be described as employing columns to support parallel main girders with joists in one direction only extending crosswise from girder to girder and a thin floor covering the joists. See Fig. 1.

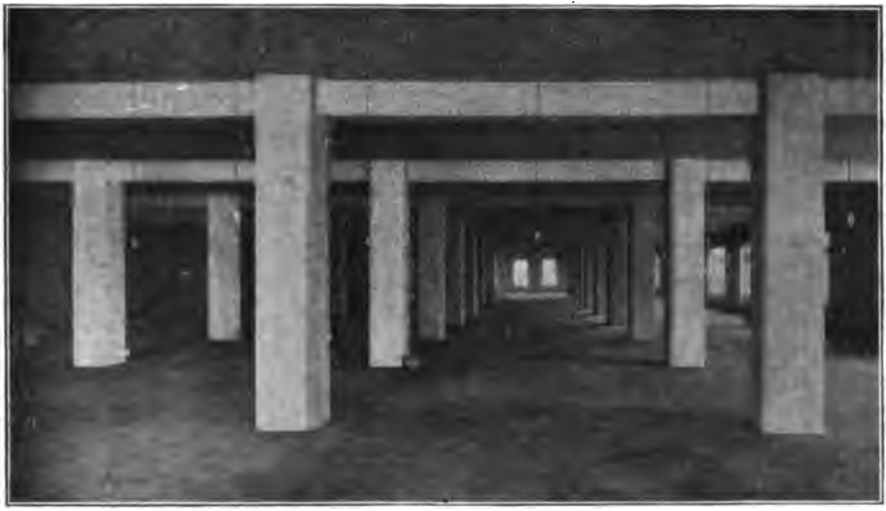


Fig. 2. Type II.

II. Similar to I, except the substitution of a slab without joists from girder to girder, similar to mill construction of beams and thick plank flooring from beam to beam. See Fig. 2.

III. A natural concrete type, a true monolith, departing from the characteristics of timber and steel construction in the employment of concrete beams from column to column in two directions and slabs with panels supported on four sides. See Fig. 3.

IV. A second distinctively concrete type, in which the centering is simplified to the limit and consists only of a temporary flooring on which to pour the concrete. The elements involved are two only: column supports, and a continuous flat slab supported directly by the columns and integral therewith. See Fig. 4.

A modification of types I and II is sometimes employed in which arches spring from girder to girder. This modification is not a common type of construction however.

As to safety, these types must be rated with reference to their deportment under overload, whether failure can occur suddenly and without warning, or slowly and gradually.

Failure is more rapid where the flexure under load has a single curvature only under load than where there is double curvature. For example, a slab supported on two sides is deformed in a cylindrical surface. The slab supported on four sides, on the other hand, dishes or bags down from all directions and cannot fail suddenly for this reason.

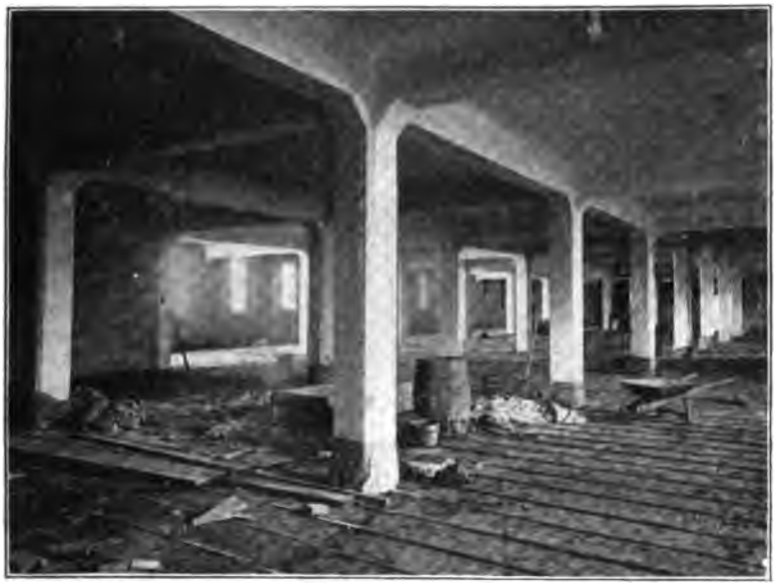


Fig. 3. Type III.

Ample lap of reinforcement over the supports, thoroly tying the work together, enhances the safety of all types.

Most failures entailing loss of life have occurred with reinforcement in one direction only and of these failures the greater part of them have occurred where insufficient lap of reinforcement has been provided.

From the fireproof standpoint, that form which exposes the least area to heat, which presents the most uniform distribution of metal to provide for the temperature stresses resulting from unequal heating will rank first, and on this basis the natural concrete types III and IV consequently are to be preferred.

In the above types, we have the following five problems in design:

(*a*) Beams, simple, continuous, partially continuous, etc.

(*b*) Slab with panels supported on two sides.

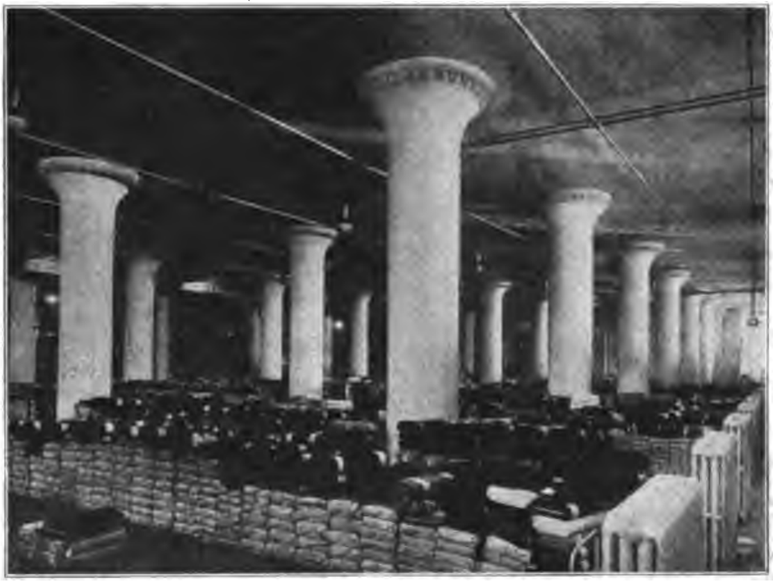


Fig. 4. Type IV.

(*c*) Slab with panels supported on four sides.

(*d*) Slab with panels supported on four posts or corners.

In each of the slab problems we also must consider the conditions of the ends or edges of the panels as in the treatment of beams.

(*e*) Columns similar for all types.

The relative economy of the several types will appear from the methods of computation to follow.

2. Utility of the Theory of Action of Structures—No theory can be devised which will take into consideration all of the phenomena presented by an actual structure. In structural work it is usual to treat for example, beams connected by flange angles or resting on top of girders as simple beams. The stresses in such beams, however, differ somewhat from what they would be on the theoretical assumption that the supports are knife edge bearings without friction. The useful theory then is that which takes into consideration the predominant phenomena presented by the structure under load. The unnecessary refinement of taking into account small, subsidiary actions, such as the restraint of connection angles, and restraint of beams which are not supported by frictionless knife edges, is for practical purposes ignored.

Thus, the direct tensile resistance of the concrete as a tension chord being small, is disregarded in practical computations. In reinforced concrete beams the tensile flange resistance offered by the steel alone as a flange is that considered and counted upon for safety as the predominant action.

The theory of work has this striking advantage over other methods of analysis of such structures, that it indicates this predominant action almost at once in a manner so clear that it requires little or no computation to arrive at a correct method of treating the structure under discussion.

3. **Principle of Proportion**—In the early development of reinforced concrete work, constructors were obliged to experiment in order to ascertain the most suitable proportions and arrangement of materials for supporting a given load on a given span, and from experiments of this kind determine by proportion the carrying strength for other loads and other spans. This principle of proportion is indeed a most useful one, and was employed largely by the builders in the middle ages in the construction of masonry work in the form of arches in the great cathedrals which command our admiration today, which work is not excelled by modern constructors with advanced knowledge of mathematics and mechanics.

The law of proportion, as applied to a slab or beam of reinforced concrete, may be stated as follows:

4. **Variation in Strength with Thickness**—It is known from elementary principles that for a given percentage of steel and a given arrangement of reinforcement, the strength of a slab or beam increases directly as the square of the depth, and for small differences in the percentage of steel, as the product of the steel area times the depth, providing, of course, the steel is not increased to such an extent that the steel element is stronger than the concrete element.

In the combination of concrete and steel, it should be observed that as between the two elements, the steel is more homogeneous, more reliable and dependable from the standpoint of uniformity of strength. Hence, the combination should be made in such manner that should failure occur it would occur in the steel and not in the concrete, and when this principle of design is followed out the reliability and safety of the structure depends on the steel element, and hence no greater factor or margin of safety is needed than in structural steel work. In fact there would be less uncertainty in this case if this principle were carried out than in structural steel work, for the reason that in structural work the members are cut by rivet holes and there is less dependence to be placed upon the large steel shapes so worked and cut than in the case of rods of uniform section and not so nicked or cut.

5. Variation in Strength with Span—The strength decreases inversely as the span for the same total load *W*, and the same moment of resistance of steel and concrete. If the strength is to be compared on the basis of a unit load per foot of span length then for the same unit load the strength decreases inversely as the square of the span.

These fundamental principles of proportion enabled the earlier constructors Coignet and Hennebique, to build successfully before the development of the theory involving the relation of the elastic properties of the two materials, concrete and steel, and enabled Turner to successfully build many great structures on the Mushroom flat slab system, prior to the development of a rational theory based upon the elastic properties of the materials. It enabled him, not only to guarantee the strength, but also to guarantee the deflection of his structures under load.

The law of proportion as to deflection may be stated as follows:

Within practical limits, including proper percentages of steel, the deflection for a given load *W* increases as the cube of the span and decreases inversely as the product of the steel area times the square of the depth from the center of the steel to the top of the slab or beam.

These proportionate relations are sufficient to enable the practical constructor, having exact knowledge of the tested strength and deportment of a reinforced beam or slab of a given design, to design a similar beam or slab for a larger or smaller load or span with certainty as to the result which can be obtained with the same grade of workmanship and the same kind of concrete.

The method of proportion applied to full sized structures has this advantage over all other methods. It takes for its foundation or starting point, the tested strength of a member approaching in size that which it is proposed to construct and a comparison is made involving a narrower range for the application of the principle of proportionality than is possible where the theorist undertakes to develop from the elastic properties of a minute sample or unit cube of the materials employed the properties of a full sized structure made of these materials. On the other hand the method of proportion has the distinct disadvantage of limitation in its scope. It cannot be applied to any form of structure which differs except within narrow limits from the proportions of the specimen with which it is compared, and hence this method is defective as compared with a general solution which enables broad conclusions to be drawn as to the generic type under consideration rather than limited conclusions specific to one form only of the genus.

The method of proportion, based as it is on elementary relations, may be used very advantageously to verify the accuracy of more complex and scientific methods of analysis. The relations above outlined follow at once from the fundamental principles governing the strength and flexure of beams and were developed in substantially the following manner by Turner in his treatise on Concrete Steel Construction, published in 1909.

6. Theoretical Treatment—A slab or beam supported at intervals either on points or on walls, if loaded, deflects or bends, and if the load is excessive the concrete cracks first from the lower or tension side upward in a plane normal to lines joining the supports. Since the reinforcing metal acts by tension along its length, it is evident in general that so far as the steel is concerned, whether the reinforcement is in single layers or in multiple layers, the action must be similar in character to the flanges of a beam, and hence the strength of the beam or slab, regardless of the distribution of the stress in the concrete, depends on the tensile stress in the steel. The mathematical expression for deflection and bending would be of identically similar form to those for beams.

W = the total load on the beam or slab, taken for convenience in thousand pound units.

L = the span in feet.

d = the distance from the top of the concrete to the center of steel in inches.

A_s = area of one reinforcing rod in square inches.

f_s = the intensity of actual stress in the steel in thousand pound units.

Σ = the usual sign of summation.

M_1 = moment of resistance for stress in the steel.

$\triangle$ = deflection at the center of beam or slab for any load.

(B) = a coefficient which may be variable or constant in value, to be so determined experimentally that $(B)\ WL$ shall equal M_1.

(D) = a coefficient similarly obtained for deflection formula.

Then by the laws of beams, we have the following equations:

$$M_1 = (B)\ W L = \frac{1}{12}\ .85\ d\ f_s\ \Sigma A_s \qquad (1)$$

$$\triangle = (D) \frac{W L^3}{\Sigma A_s d^2} \qquad (2)$$

In formula (1) it will be noted that $.85d$ is assumed as a close approximation to the effective lever arm of the steel jd, or the distance from the centroid of tension to the centroid of compression in the beam or slab. This, of course, varies slightly with different percentages of steel, but for practical purposes it may be assumed that this value does not involve material error, and is on the safe side.

The coefficient (B) for the simple beam is $\frac{1}{8}$. For the continuous beam it is customary to take this as $1/12$ at the support and $1/16$ at mid span, while for the slab supported at four sides, (B) is taken as the reciprocal of 30, and where the reinforcing metal is more closely spaced at the center third, the reciprocal of 36, while (B) for a mushroom slab is taken as the reciprocal of 50. It is assumed in these formulas that $f_s = 13$, which is expressed in thousand pound units.

The deflection $\triangle$ of slabs will follow the same laws as the deflection of beams so far as factors depending upon W, L, and I are concerned, and will consequently be equal to some multiple of WL^3/EI. But I varies as $\Sigma A_s d^2$. Hence $\triangle$ varies as $WL^3 / \Sigma A_s d^2$, which is expressed above in equation (2) in which the constant multiplier (D) is to be determined experimentally, but could supposedly be derived analytically in case a sufficiently general theory should be developed to express correctly the manner in which it depends upon the known arrangement and properties of the materials composing the slab.

For simple beams, (D) is taken as the reciprocal of 850 and for continuous beams cast integrally with a heavy slab, as the reciprocal of 5,000.

(D) is taken as the reciprocal of 10,000 in the slab supported on four sides, and for the Mushroom system as the reciprocal of 7,000.

The application of the principle of proportion in the determination of the working stresses is based on the assumption (which is on the side of safety) that steel stress under working load is proportional to the steel stress at the elastic limit of the steel under a load which would produce this stress. Now the elastic limit of medium open hearth steel has a practically fixed value which varies little and which can readily be determined by test.

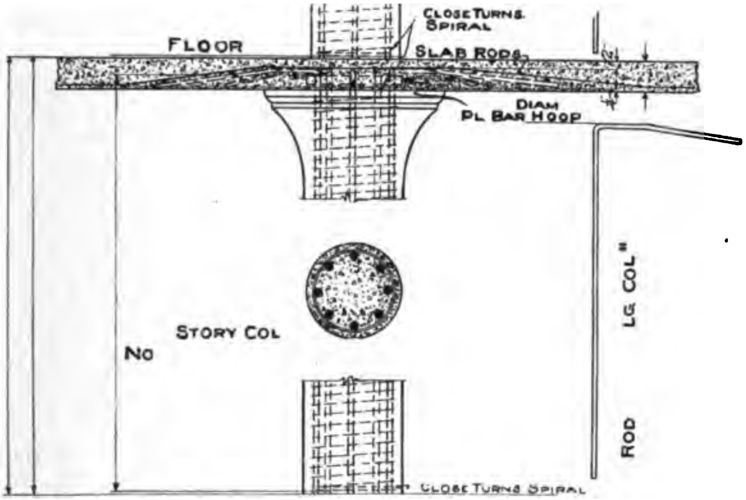


Fig. 5. Vertical Section of original Mushroom Head showing position of Radial and Ring Rods, and Slab Rods, Vertical and Horizontal Sections of Spirally Hooped Column, with Plain Bar Hoop Collar Band, Vertical Reinforcing Rods and Elbow Rods.

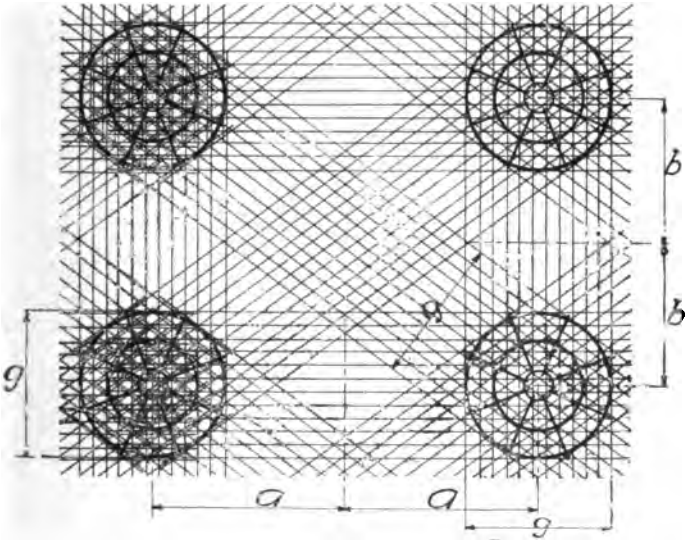


Fig. 6. Plan of Reinforcement in original Mushroom System. Radial and Ring Rods, Collar line not paralled to other lines, Band and Slab Rods. Diameter of Head $= g = \frac{7}{16}(a+b)$.

Having constructed a panel or beam reinforced with this known grade of metal, by loading until the first yielding of the steel occurs, we then know the steel stress under the applied load, and may, by proportion, determine closely the steel tension for any smaller working load.

The elastic limit above referred to is the limit of elasticity of shape as determined by a slowly applied load, and is to be distinguished from rigid proportionality of shape. By this method of investigation, coefficients (*B*) may be determined for all types of construction. Coefficients (*D*) are figured from the measured deflection of the member or panel tested. This method of determining the coefficients used is, of course, limited to those designs in which the rational method is followed of proportioning the structure so that the steel element determines the safety of the structure.

No formulas for strength, based on the elastic properties of the materials can be accepted as correct unless the corresponding formula for deflection can be depended upon also. In other words, an elastic theory in order to be acceptable must demonstrate its accuracy by agreement of the entire elastic deportment of the structure to which it is intended to be applied including both stresses and deflections. The determination of the respective coefficients for strength and deflection having been derived independently of each other, empirically or by experiment, their general agreement can be established by a concordance of the deflections observed with those computed in structures which have been designed for strength, using the same coefficients (*B*), for both.

We have noted the method of determining the coefficients (*B*) and (*D*) for a specific type of construction. These coefficients can be used for any size or thickness of panel or percentage of reinforcement if they remain constant in value and are not variables. Their values may be rendered constant by fixing the arrangement of reinforcement in strict proportion to the sample tested. We will illustrate this proposition by its application to the Mushroom type of reinforcement shown in Figs. 5 and 6. In this construction the values of the coefficients are made constant by fixing the diameter of the Mushroom head and width of belt as identically or approximately the same fraction of the span of the test panel from which the coefficient was determined. In other words, if the diameter of the head and corresponding width of belt be kept within the limits of 7/16 to 1/2 the distance between column centers in the case of square panels, or 7/32 to 1/4 the sum of the long and short spacing in a rectangular panel, then the coefficients remain practically constant. Otherwise, they become extremely variable, increasing in value several hundred percent as the width of belt is reduced to 50 percent of the above proportions. From this statement it becomes evident that coefficients of this character must be applied rigidly to similarly proportioned types of reinforcement until the law of their variations is accurately determined.

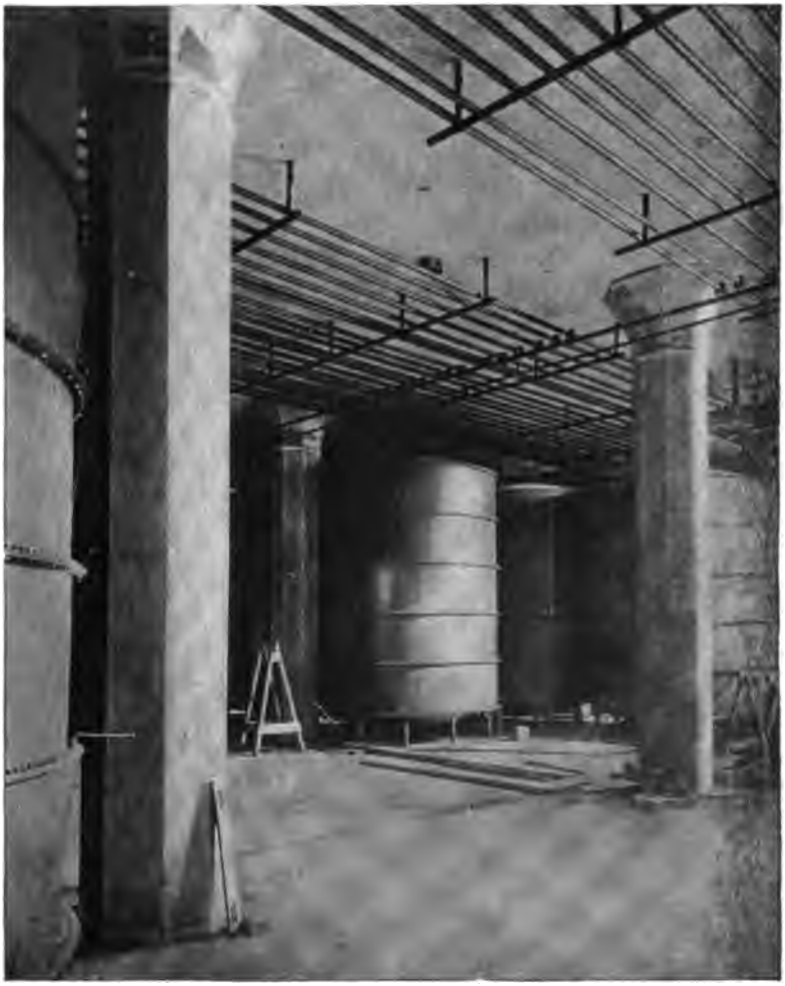


Fig. 7. Interior Stock House Hamm Brewing Company Building, showing 50 ton tanks being placed in position covering full area of floor.

L in the formulas for bending and deflection is taken as the longer direct distance between column centers, the shorter direct distance appearing in the case of a rectangular panel in the determination of the diameter of the head and in the determination of the load *W* from the unit load per square foot.

In the treatment of the steel area it may be noted that the line of weakest section at failure as determined experimentally, cuts across the four way belts practically at right angles. Hence the resisting moment is taken at this, the weakest section, as the elastic limit of the steel is approached, although the maximum stress in the steel does not occur actually at this section on the diagonal belts under lesser load.

7. **Example**—An illustration of the application of these formulas is given herewith for the Mushroom system in the floor of the Hamm Brewing Company Stock house, illustrated in Fig. 7.

Take the case of a panel of the Hamm Brewing Company's building, shown in Fig. 7, panel 22′10″ by 26′0″ loaded with four tanks 10′ in diameter, 15′ high full of water. The load is equivalent to 200 tons of uniformly distributed load. The floor slab is 14″ thick at the outer edge and pitches upwards 3″ to the center and is reinforced with twenty-five ⅝″ rounds each way. Taking an equivalent depth of 15¼″ as the distance from center of steel to top, we have the following equation:

$$\Delta = \frac{1}{7000} \times \frac{(400)\ (26)^3}{(4 \times 25 \times .3)(15.25)^2} = .144''$$

Another panel in the same building, 20′10″ by 20′8″. Same loading, thickness and reinforcement.

$$\Delta = \frac{1}{7000} \times \frac{(400)\ (20.83)^3}{(4 \times 25 \times .3)(15.25)^2} = \text{a full } 1/16''.$$

These figured deflections proved exactly equal to the measured deflections as nearly as the engineer of the brewery could determine by marking the same with a knife edge.

We will take another case. Test of the State Factory Building at Stillwater, Minn., Mr. C. H. Johnston, Architect. Size of panel 19′9″ by 20′8″. Thickness of slab 8″. Reinforcement seventeen ⅜″ rounds each way. Test load, 450 lbs. per square foot over the full area.

$$\Delta = \frac{(180)(20.66)^3}{(7000)(4 \times 17 \times .11)(7.25)^2} = .573'' = 9/16''$$

the reported deflection.

The Hoffman Building, Milwaukee. Test load 142 tons. Panel 17′0″ by 16′8″. Reinforcement seventeen ⅜″ rounds each way. Slab 8½″ (7½″ rough and 1″ finish).

$$\Delta = \frac{(284)\ (17)^3}{(7000)\ (4 \times 17 \times .11)(7.8)^2} = .437 = 7/16''$$

the measured deflection.

Another example: Test of the John Deere Plow Company's building in Omaha. Fig. 8. Panel 18′9″ square. Reinforcement sixteen ⅜″ rounds diagonally and fourteen ⅜″ rounds directly from column to column. 7″ slab in rough, with strip fill added later about 2¼″ thick and ⅞″ finish floor of maple. This we find an equivalent to a slab of about 8¾″ concrete as far as deflection is concerned, the strip being a 1:3⅓:4·mixture.

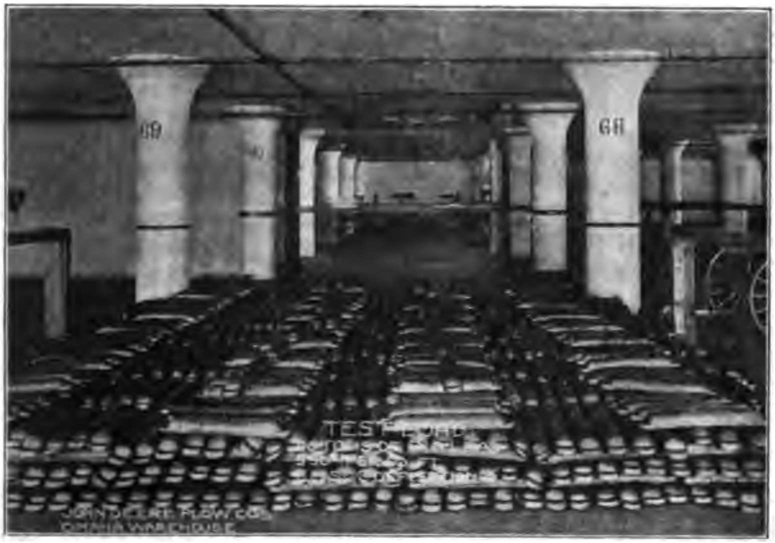


Fig. 8. Test of John Deere Plow Company Building, Omaha, Neb.
550 pounds per square foot.

$$\Delta = \frac{(160)(18.75)^3}{(7000)\ (6.6)\ (8)^2} = .356'' \text{ or } \tfrac{3}{8}'', \text{ the measured}$$

deflection.

8. Slab Supported on Girders—The treatment of a rectangular slab supported on four sides and cast integrally with the supporting beams by the method of proportion will also be illustrated.

The common arrangement of such slabs is shown in Figs. 9 and 10. For square slab the load, of course, is divided equally between the four supporting beams and coefficient (B) is taken as the reciprocal of 30 where the rod spacing is uniform, and as the reciprocal of 36 where the rods are spaced twice as closely for the middle third as they are for the outer third. The coefficient (D) is taken as the reciprocal of 10,000 for both types. In the case of a rectangular panel on the assumption that the load transferred to the beams is in proportion to the lengths of the sides, a and b, a mean length for moment and deflection is derived by taking the quotient $(a^2+b^2)/(a+b)$ so that the formulas become:

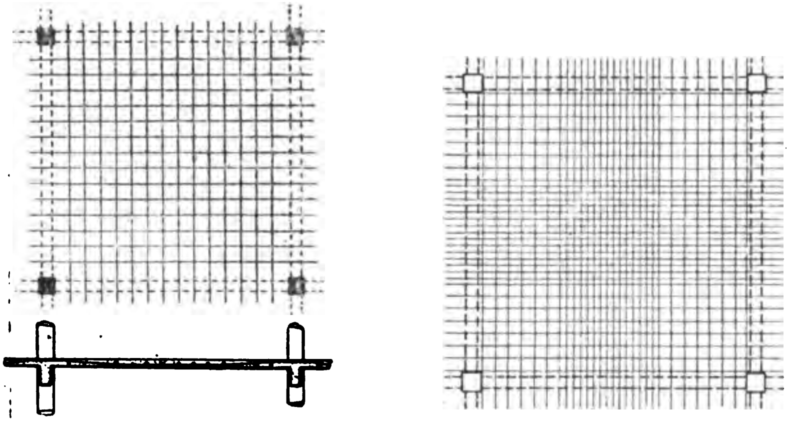


Fig. 9. Fig. 10.

Common Type Slabs, supported on four sides, reinforced two ways.

$$M_1=(B)\ W(a^2+b^2)/(a+b)=.85\ d\ f_s\ \Sigma\ A_s/12$$

$$\text{and } \Delta=(D)\frac{W\ (a^2+b^2)^3}{(a+b)^3\ \Sigma A_s\ d^2}$$

9. Computation of Deflection Applied to Practical Examples—We will now proceed to apply these formulas to two cases:

First, take the Minneapolis Paper Company's building, details shown in Fig. 11, photograph of test load in Fig. 12. Slab 7″ in the rough, 15′4″ by 21′6″ center to center of columns, strip fill $1\frac{3}{4}$″ and $\frac{7}{8}$″ finished floor. Now taking the strip fill as effective for one half of its actual thickness, we find for a load of 234,000 lbs:

$$\Delta=\frac{1}{10{,}000}\times 234\times\frac{\left(\dfrac{21.5^2+15.3^2}{21.5+15.3}\right)^3}{(75\times 11)\times(8)^2}=.30''$$

= deflection of slab at center as measured less the beam deflection.

Take for example the test made at the Smythe block at Wichita, Kans. Test load consisted of fifteen tons concentrated at the center over an area of 7 feet square equivalent to about 45,000 lbs. uniform load. Size of slab 20′9″ by 24′9″, 6½″ in the rough, 1¾″ strip fill. Reinforcement, ⅜″ rounds 5″ centers for the central third of the panel each way and 8″ centers outside third width each way. Then—

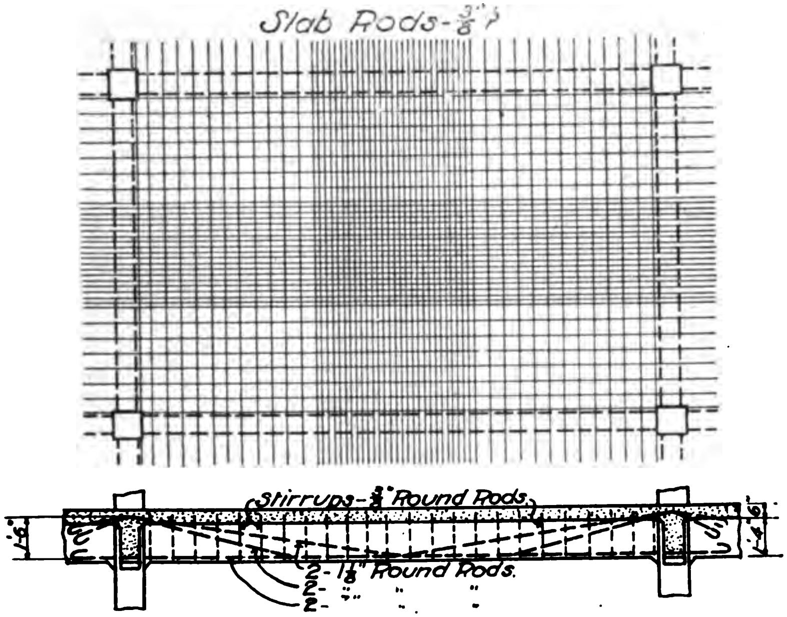


Fig. 11. Details of Reinforcement in Panel of Minneapolis Paper Company Building.

$$\Delta = \frac{1}{10{,}000} \times (45) \; \frac{\left(\dfrac{20.75^2 \times 24.75^2}{45.5}\right)^3}{9. \times 7.25^2} = .11'' \text{ or}$$

a full 3/32″ the deflection measured.

In the test at Wichita, the beam deflection, owing to the small load on the panel was practically negligible, and does not need to be considered in arriving at the true slab deflection.

Test of the Minneapolis Knitting Co.'s Building:

Slab 5½″ thick with 1¾″ strip fill, panel 16′4″ x 15′8″. Reinforcement ⅜″ round rods 4″ centers each way.

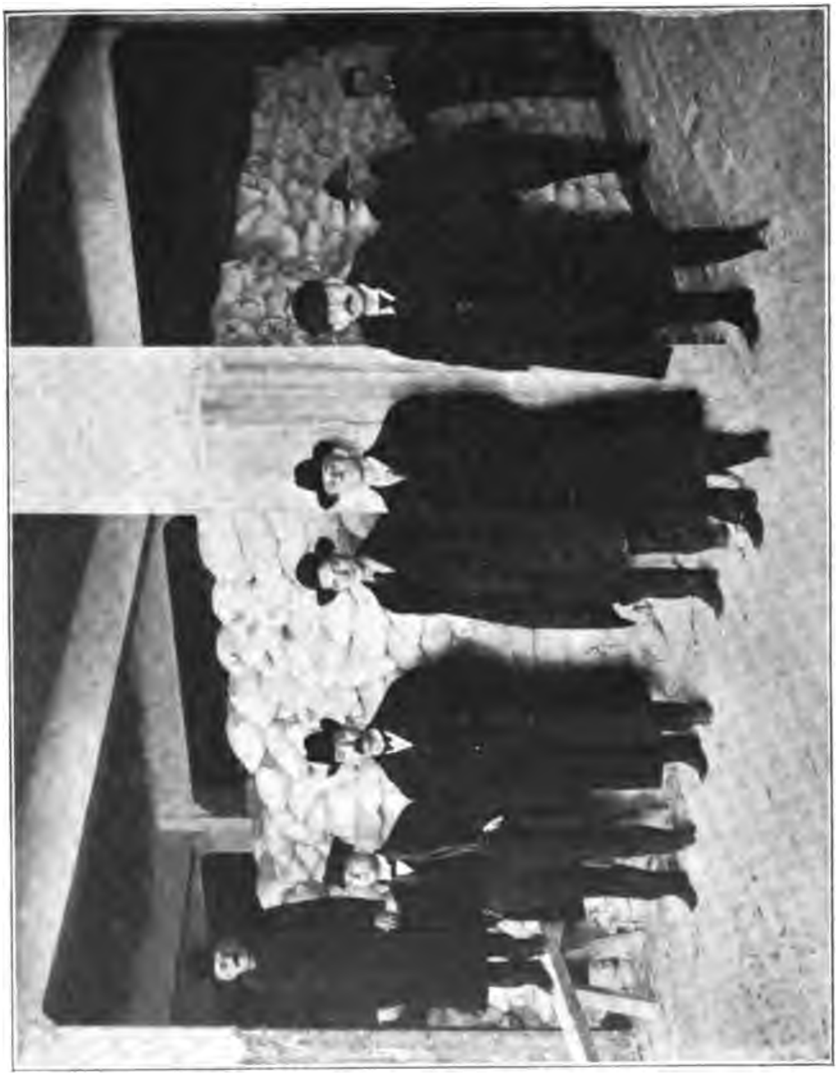


Fig. 12. Photograph of Test Load, Minneapolis Paper Company Building.

$$\Delta = \frac{1}{10500} \times 160 \times \frac{\left(\frac{16.33^2 + 15.66^2}{32}\right)^3}{(10 \times .4)(6)^2} = .167'' = 5/32''+.$$

The measured deflection agreed identically with that figured.

This formula is based on the assumption that a large fraction of the internal work of deformation is performed by lateral action after the manner of a uniform continuous plate and while not strictly accurate it is in much closer approximation to the actual condition of stress than irrational formulae based upon a distribution of stress and load in proportion to the fourth power of the respective sides as derived in the irrational treatment on the basis of independent beam strips through the center of the panel, or upon an inapplicable approximate solution of the general differential equation of such a slab as derived by Grashof. The agreement of the formula for deflection gives closely approximate results from the practical standpoint and its mathematical deviation from correct values will be discussed later.

The method of design by proportion, based on the steel stresses as explained in the preceding pages, presupposes that the concrete element resisting compression is of greater strength than the tensional steel element as it should be in conservative design. The limiting steel percentages for the various types of structure will be discussed later rather than under this present heading.

10. Short Span Slabs and Arch Action that may be Counted upon in Their Use.—We have heretofore discussed at some length long span slabs. A long span slab will be defined as a slab, the ratio of whose thickness to length is so small that the possibility of its acting effectively as an arch is eliminated.

It is to long span slabs where the arch action is negligible that formulas for bending apply with a high degree of precision. When, however, we come to test a short span slab which is part of a continuous floor there may be quite a large amount of arching in the slab by which the load is carried to the support without causing tension in the steel to the extent that it would do provided there was no rigid skew-back to sustain the thrust. Where the span of the slab does not exceed ten times its thickness it is permissible and good practice to figure the bending moment on the slab at $WL/16$, and this moment is to be increased to $WL/10$ where the thickness is one-sixteenth of the span, with intermediate values for intermediate ratios of thickness to span. Where the span is more than sixteen thicknesses of slab it is to be figured as heretofore provided.

11. Value of Finish Coat, Strip Fill and Wood Floor from the Standpoint of Deflection—A slab the depth of which has been increased twenty percent by a strip filling of concrete having considerable rigidity or ten percent of its depth with a wearing coat if figured on the basis of the actual thickness of the rough slab, presents a surprising degree of stiffness, but a more conservative computation taking into consideration the actual value of the strip fill as about half of the normal added thickness will give results in close accord with measured deflections.

Where added stiffness or strength is an object, the same grade of concrete should be used in the strip fill as in the slab, instead of weak indifferent filling of cinder, natural cement or Brown Lime concrete which some employ. A strip fill of Protland Cement concrete hardens and dries out more rapidly than one of natural cement or Portalnd cement and lime mortar, and permits the finished floor to be laid at an earlier date without danger of having the hard wood swell, buckle and pull away from the cleats to which it has been nailed by reason of the moisture absorbed from the uncured filling. Trouble from this cause is common in the Fall of the year, whenever lime is used in the fill, the author having noted the lateral swelling of kiln dried Maple to the extent of fifteen inches in a width of 50 feet. Six widths had to be taken out of the floor and the floor taken up and entirely relaid. A saving in the first cost in the cement and lime mortar fill thus proved quite expensive in the end.

Where dependence is placed on the strip filling for stiffness or strength, the concrete rough slab should be cleaned with a steel brush, given a neat coat of cement grout, immediately before adding the finish coat or laying the strips and strip filling. Such a strip fill 1¾″ thick at the thinnest place with strips 16″ on centers and ⅞″ wood floor nailed thereto generally deflects about the same as though an inch and a quarter finish coat had been applied to the rough slab of concrete. Such a strip fill, however, will usually exceed the nominal thickness of one and three quarters inches since it is impractical to leave the top surface of the rough slab perfectly level and as a rule the actual thickness of a normal 1¾″ fill will not be less than 2¼″ in the center of the slab though it may be less around the columns or beams.

12. The Maximum Moment and Deflections in Flat Slabs and in Beams Compared—The maximum bending stresses developed in structures having equal spans and equal cross sections are proportional to (B), and the maximum deflections are proportional to (D), while the strength and stiffness are proportional to the reciporcals of (B) and (D) respectively. For example, we have by the common theory of flexure in accordance with Chapter II, the following Comparison of homogeneous beams of span $= L$, with a total uniform load $= W$.

Kind of Beam	Maximum Moment $M = (B)\,WL$	Relative Strength	Maximum Deflection $\Delta = (D)\dfrac{WL^3}{EI}$	Relative Stiffness
	(B)		(D)	
Simple End Supports.........	1/8	1	5/384	1
Fixed at one end supported at other end..............	1/8	1	1/185	2½
Fixed at both ends (*a*) at middle	1/24	mean, 2	1/384	5
Fixed at both ends (*b*) at ends	1/12			

It thus appears that a uniform beam uniformly loaded is about twice as strong when quite rigidly fixed at its ends as when freely supported and five times as stiff.

Now steel stresses in reinforced concrete beams under heavy working and test loads show acting moments of 60 to 90 percent or more of the above values of the external or applied moments in homogeneous beams.

Of the four types of reinforced concrete floor structures considered in Chapter II, as has already been noted, Types I and II imitate older types of timber construction in external form, while III and IV are distinctive concrete types differing from timber and steel structures in being true monoliths in form and mode of operation. The notable difference between these latter two types and the first two is the remarkable strength and stiffness of III and IV which is brought out strongly by the smaller relative magnitude of the coefficients (B) and (D) which have been developed by test as stated in Chapter II.

This is brought out more clearly by a tabular comparison of the experimental values given below for square panels of reinforced concrete slabs of span $= L$ and uniformly loaded with a total load $= W$, as follows:

Kind of Slab	Maximum Moment at mid span $M=(B)WL$	Relative Strength	Deflection at Panel Center $D=(D)\frac{WL^3}{EI}$	Relative Stiffness
	(B)		(D)	
Simple supports at 2 opposite sides..............	1/8 to 1/10	1	1/850	1
Continuous over 2 parallel panels or monolithic beams................	1/12 to 1/24	mean 2	1/5000	6
Panel supported at 4 edges on walls or stiff beams				
(a) rods equally spaced....	1/30	mean 3	1/10 000	12
In middle third...........	1/36	mean 4	....	....
Continuous Mushroom slab with monolithic column supports Moment at center......	1/50	In terms of the diagonal span	$1/7000\sqrt{2}$	11.2
		In terms of the direct span	1/7000	$\frac{11.2}{\sqrt{2}}$

Before entering upon any mathematical analysis of the several types of floor structures of reinforced concrete it will be useful to consider somewhat their relative strength and stiffness as revealed by the values of the experimental coefficients (B) and (D) stated in the Table just given.

A continuous beam of reinforced concrete, whose depth is about one tenth of its span, which is monolithic with its supporting columns, appears from the coefficients in Chapter II deduced from tests to be able to carry double the working load of a simple concrete beam of similar dimensions and cross section of reinforcement, and to be six times as stiff.

This somewhat greater relative stiffness of the fixed concrete beam as compared with one of homogeneous material may perhaps be accounted for by increased arch action made possible by the large ratio of depth to span in the concrete beam. The resistance may thus be divided between predominant beam action and subsidiary arch action in the proportion roughly of 80 or 85 percent for beam action and 15 or 20 percent arch action, a distribution between the actions which transmit the load to the supports which leaves but a negligible proportion of the resisting moment to be accounted for by the tensile strength of plain concrete, as would be expected upon the hypothesis of negligible tensile resistance of concrete which is universally adopted by practical constructors.

Consider now the stiffness of a rectangular plate simply supported on two opposite sides as compared with the stiffness of a similar plate supported on four sides and fixed or continuous over the supports, such as that included in Type III. Comparison of the coefficients show that such a plate is nearly 12 times as stiff as the same plate simply supported on two sides instead of only five to six times, as might be expected from the relative stiffness of simple and continuous beams. It should be noted, further, in the case of a flat slab structure of Type III, and Type IV, also that the small ratio of thickness to span, viz., from $L/25$ to $L/40$, precludes arch action of any such magnitude as that in continuous beams having a depth as great as $L/10$.

It may be observed that slabs of Class II and beams and joists of Class I, are bent in cylindrical curvature about one single axis only while the distinctive peculiarity of the bending of slabs of Class III and of IV as well, is that of dish shaped curvature or bending about two axes instead of one only.

An even more marked increase in stiffness than that exhibited by slabs of Type III is found in interior square panels of the original Mushroom Type Class IV. The measure of stiffness or coefficient of deflection given in the previous Chapter is based upon consideration of the direct span instead of the diagonal span upon which the deflection figured is actually measured.

If we replace L^3 in the numerator of formula (2) page 36, by $(L\sqrt{2})^3$ to reduce it to terms of diagonal span and also multiply the denominator coefficient by $2\sqrt{2}$ to make it correspond with the numerator, the value of the denominator of (d) will become, when referred to the diagonal span of Mushroom slabs, $7000 \times 2\sqrt{2} = 19\,769$ instead of 7000.

Thus the relative stiffness of an interior panel of a column supported Mushroom flat plate is as 19 679 to 850, or more than 23 times as great as that of simple beam or a simply supported rectangular plate under the same load when the span is taken as equal to the diagonal of the panel with the same thickness of concrete and cross section of reinforcing steel at mid span.

The deflection at mid span on direct lines from column to column of any square column-supported panel of the original Mushroom type is approximately half as great as the deflection of the diagonal center of the panel. Comparing this deflection with that of the continuous slab of the same thickness and same reinforcement simply supported on parallel girders the deflection on direct column lines would consequently be one twenty-third that of the simply supported plate.

Again, comparing it with the deflection of the same slab when integral with these girders and continuous over them, its stiffness is nearly five times as great as such a slab on direct column lines and two and one half times as great between parallel column lines.

Although floors of Types III and IV, by reason of their distinctive monolithic character and manner of bending, possess elements of strength not found in the beam type, nevertheless, throughout the literature of reinforced concrete, and in building code rules as well, the attempt has been made to analyze these types by regarding monolithic slabs for the purpose of calculation as separated into strips comparable to ordinary beam elements and treating these strips by a mere adaptation of beam theory.

This attempt to treat monolithic plates which are bent in double or dish shaped curvature on mere beam theory errs in the first place in that it disregards the fundamental statical principle that where a slab is column supported, the external bending moment due to the loads and the supports, is inevitably greater than the internal moment of resistance of the longitudinal fibers due to bending because the external applied forces produce at the same time both bending and twisting moments, and not bending only as has been heretofore erroneously assumed to be the case. The sum of the bending and twisting moments taken together is equal to the sum of the moments of the applied external forces. Thus when the deformation of the plate is cylindrical or the curvature is in one direction only, there is no twisting effect at all, and the only stresses called into action in bending are those of longitudinal resistance of the fibers. When, however, the curvature is more complex, such as occurs in dish shaped curvature, in the column supported flat slab with the saddle shaped areas along direct lines, the concave, convex curvatures combined produced by loading, we have the condition of combined bending and twisting.

This attempt errs in the second place because it disregards the fundamental principles of the deformation by bending of thin convex or concave plates, which principles apply to the dish shaped areas around panel centers and around supports. These principles as developed by Kelvin* and by Love† show that it is impossible to

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*Thomson & Tait's Natural Philosophy.
†Love's Math. Theory of Elasticity.

bend a nearly flat dish shaped shell about one horizontal axis without at the same time bending it in the opposite direction about a second horizontal axis at right angles to the first. The beam strip theory which disregards this fundamental fact is necessarily erroneous.

13. **For the reasons just stated,** it is not at all surprising to find an irreconcilable discrepancy between the results of measured steel stresses in floor slabs, Types III and IV and calculations of the magnitude of these stresses on beam strip theory. This wide discrepancy is conceded by all. For example, Mr. Hatt, p. 1560 Transaction of the American Society of Civil Engineers for 1918, Vol. LXXXII, says:

> "Various tests of flat slab buildings, as reported in the Bulletin 84 of the Engineering Experiment Station of the University of Illinois, and elsewhere, are grouped in Table 8.

Part of Table 8—From Transactions American Soc. Civil Engineers

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Results of Tests of Flat Slab Floors

Moment Coefficients n where $M=nWL$. Accounted for by Measured Steel Stresses.

Building	Load in Lbs. per sq. ft.		Number panels loaded	Positive		Negative		Total Coeff.	Percentage	
	Design	Test		Stress	Coeff.	Stress	Coeff.		Theory	Joint Committee
(1)Franks*.	250	739	2	10 100	.0068	9 200	.0149	.0217	24.0	28.0
(2)Larkin*.	250	738	2	16 000	.0090	8 500	.0135	.0225	25.0	29.5
(3)Shultz †.	300	844	4	6 200	.0038	7 100	.0105	.0143	15.8	19.0
Average..	.....	.....	....	......	.0065	.....	.0130	.0195	21.6	25.5

†Identical with or * similar to original Mushroom Type.

> Tests of continuous beams and girders of existing buildings are reported in the Bulletin of the Engineering Experiment Station of the University of Illinois, as follows:

	Steel Stress	Percentage of load moment in steel stresses
Beam	16 000	77.0
Girder	17 000	79.0
Beam	7 500	64.0

> "From an inspection. . . . of Table 8, it is evident that even at steel stresses where the concrete is broken down in tension at some sections, there is a marked* descrepancy between internal and external moments."
>
> *(Which amounts to three or four hundred percent according to Hatt's figures).

The fact that this resistance persists after the concrete has cracked or broken down at various sections and in this condition balances from three fourths to four fifths the external moment of the load *when* the separated sections are connected by reinforcing elements *only*, and *only when* the slab is bent in dish shaped curvature, indicates clearly that this resistance is the product of a distinctive kind of resistance brought into action in this particular class of slabs by their distinctive manner of bending. Such action is foreign to beam action, differing widely therefrom in the quantitative phases of carrying capacity and stiffness under load.

14. A Common Objection to the formulae derived by the principle of proportion and confirmed by the results of hundreds of tests of successful floors is that where it has been attempted to apply these formulas to conditions of restraint at the support wholly different from those in the tested structure from which the formulae were derived they do not correctly give the resulting strength or stiffness. This objection is as illogical as it would be to object to the accuracy of the mathematical formula for deflection and strength of restrained beams on the ground that if the restraint is removed this accurate formula no longer applies to the simple beam.

It is true that when the formulae deduced in the preceding chapter have been applied to similar slabs supported by columns of various diameters, length and reinforcement, some with a large increase in stiffness and rigidity of the column over and above a certain minimum, a relatively small increase in the stiffness of the floor results. A very considerable decrease in the strength and stiffness may on the other hand result from a reduction of the degree of restraint and rigidity of the supports below the minimum to which the formulae for proportion apply and the conditions of rigidity under which they were derived.

In an interior panel where a floor is uniformly loaded the stiffness of the column under flexure is not brought into the problem, for as long as the column is sufficiently strong to take the vertical load under these conditions, the slab is to all intents and purposes fixed. There are thus presented for consideration two entirely different problems: First; the analysis of the action of the floor slab as a uniformly loaded structure in which because of the symmetry of loading the reactions at the support are known or determined, and Secondly; the indeterminate problem in which because of the lack of symmetry of loading or because of unlike end conditions and restraint the reactions are unknown and dependent upon the relative rigidity of the supports and the degree of restraint imparted to the floor or beam. An analysis of the problem of the continuous beam and slab where the reactions at the supports are known, will be first presented, later the other problem will be attacked by applying to it the principle of rigidities and law of least work by which the reactions are to be determined and the indertermination resolved.

It has been frequently argued that if Mushroom floors or other floors figured in the same manner should be loaded to more than the full working load, inherent weakness would be manifest and failure follow. Now this objection to the acceptance of results of hundreds of tests is without foundation and evidently arises from lack of familiarity with the manner in which warehouses are commonly loaded. A warehouse man in renting a building, makes his profit on the tonnage of the material stored. His method of storing sugar for example is to commence at the end of the building furthest away from the elevator, pile material such as sugar from the floor to the ceiling and from one end of the building to the other, thus rendering the loading similar to a moving load. A case in which the load exceeded double the permissible live load was investigated by the author. The load had been on the floor for a period of three months and while the tenant was severely criticised for such abuse of the construction he regarded the criticism as unfounded because no serious result or apparent over strain was observed during this period.

CHAPTER III

MECHANICS OF STRESS

1. Stress. When a body is subjected to an external force, the opposing distributed internal force which resists deformation or change of shape is called stress.

If a body be supposed to be divided into two parts by an imaginary plane traversing it in any direction the intensity of the distributed force exerted between those two parts of the plane of division is an *internal stress.*

The stress upon a plane section of a body, may be defined as the equal and opposite action and reaction that the two parts of the body exert upon each other when any forces act upon the body. The stress acting at a surface is measured in pounds, per unit of area.

Elementary forces producing stress, are of three kinds: Tension, tending to elongate the piece which is stressed; compression tending to shorten it; shearing force which tends to slide two contiguous parts of a body, one upon the other, and so to produce angular deformation or shearing distortion.

When this shearing or rotational distortion exceeds the limit of the shearing elasticity of the material, sliding occurs.

In expressing tension and compression, algebraically, the positive sign will be assigned to compression and the negative sign to tension as a matter of convenience.

In a fluid the internal stress at any point is of equal intensity for all positions of the plane of division. In the solid body on the other hand, the stress may be either normal, oblique or shearing, and it may vary both in direction and intensity as the position of the plane of division varies.

The internal stresses that act on the faces of any element of a body at rest are combined on the principle of the parallelogram of forces, and taken all together they hold it in equilibrium.

2. Elasticity. Elasticity is that property of matter by virtue of which a body requires force to alter its bulk or shape, and requires a continued application of that force to maintain the alteration and also causes it to spring back and return to its original shape or form when the applied force is removed.

Every material has some degree of elasticity of bulk. If it possesses any elasticity of shape it is called a, solid. If it possesses no elasticity of shape it is called a fluid. All fluids possess perfect elasticity of bulk.

The limits of elasticity for steel and concrete are so narrow that the distance between any two neighboring points of the substance never alters more than a very small proportion of its own amount without either breaking or experiencing a permanent deformation. And therefore the angle between any two lines, meeting at any point and passing always through the same matter, is never altered by more than a small fraction of itself before permanent distortion or rupture occurs.

For steel, the elastic elongation or shortening before permanent set takes place varies from 1/700 to 1/1,000 of the length, and according to Kelvin the angular distortion cannot probably exceed half a degree before permanent set occurs.

Hooke's Law. The mathematical theory of elasticity is based on Hooke's Law, first published by Robert Hooke in 1678,—"*Ut tensio sic vis,*" "as the extension, so the resistance", or freely, "Extension is proportional to force (applied)". Hooke applied this law to all springy bodies, naming nearly all materials of construction.

Deformation, or Strain, is synonymous with the word extension as above used. From experiments on materials used in construction it has been found that when the unit stress in any elastic body does not exceed a certain limit the strain or deformation in the direction of the stress, bears substantially a constant ratio to the stress or force which produces it.

Elastic elongation due to a longitudinal stress is accompanied also by lateral contraction of the body. Similarly, shortening or compression, due to longitudinal stress is accompanied by lateral expansion, thereby increasing the cross section. These lateral deformations are also proportional to the longitudinal stress just as the longitudinal deformations are.

Strain. True strain is the elastic change in shape or deformation resulting from stress. On removal of the stress true strains disappear. The word strain is commonly used to denote deformation per unit length under the unit stress applied.

Deformation is commonly used to denote the total strain produced by the total applied force.

Deformation from tension or compression which does not disappear on removal of the applied force is sometimes called permanent strain as opposed to true strain but is more commonly called set.

Detrusion is the shearing distortion caused by shear.

3. Applied Forces—Normal or Tangential. Any distributed force applied to a body can be resolved into normal and tangential components at the surface to which it is applied. When components produce normal compression they are positive, and when they produce normal tension they are negative. Tangential components which act to produce angular rotation in a clockwise direction are considered positive and those tending to produce anti-clockwise rotation are negative.

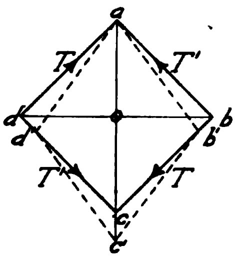


Fig. 13
Illustrating Deformation Produced by equal Tangential Forces on an element.

Suppose the equal faces of an elementary square prism *abcd* of an elastic body be acted upon by equal tangential forces *T* or *T'*, along each of the four sides in the direction of the arrows.

These forces will distort the square into a rhombus *ab'c'd'* and the total angular distortion of the area abcd will be twice the angle *bab'*.

As the original diagonal *ac* of the square prism has been lengthened to *ac'* the tangential stresses have produced tensions normal to *d'b'*. Because *db* has been shortened to *b'd'* compressions have been induced normal to *ac*. Consequently, tangential stresses produce not only angular deformation but direct tensile and compressive stress and strain as well.

Moment. The moment of any single force *T* (Fig. 13) about a point *O*, is the product of *T* by the shortest or perpendicular distance from *T* to *O*. This moment consequently depends both upon the magnitude of *T* and the position of *O* with respect to *T*. The distance of *O* from *T* is called the lever arm of *T* about *O*.

Couple. Two parallel forces $+T$ and $—T$, of equal magnitude but of opposite sign constitute a couple, and the algebraic sum of their moments is the same about any point whatever in their plane, and is called the moment of the couple. Its magnitude is the product of *T* by the perpendicular distance between the lines of action of the forces of the couple. This moment may be regarded as acting about any point whatever of the plane or any line normal to that plane.

Twist. Let any force *T* of a couple act parallel to the plane of the paper along the face *da* of an element *abcd* of a shaft which extends perpendicularly to the paper, and let the other opposing force *T* act along *bc*, then this couple furnishes a positive twisting moment tending to rotate the shaft in a clockwise direction. In case the rotation of the shaft is prevented the reaction is furnished by the internal resistance of the material, equivalent to a couple $\pm T'$ acting along the other two faces of this element *abcd* on some element parallel to *abcd* situated at some distance from *abcd* along the shaft. By such action twisting moments are transferred along the length of a shaft by means of the elastic resistance of the material. This elastic resistance consists of a twisting shear on all right sections of the shaft parallel to the face *abcd*. Thus twist produced by moment transfers shear and moment along an axis of moments, in this case along the length of the horizontal shaft transmitting power to any desired point, and the moment remains unchanged from point to point until a pulley is reached which reduces the moment by transmitting power from the shaft to a machine.

The Direction of the Axis of Moments may be defined as any line drawn perpendicular to the plane containing both the force and the lever arm. Consequently, **Moment is a Directed Quantity,** acting about an axis fixed in direction. Its amount may be laid off along that axis to represent it graphically just as a force may be laid off along its line of action. It can only be combined with the moments of other couples by combining their axes according to the parallelogram law in the same manner as forces are combined with each other.

For example, consider the case of equal positive bending moments acting about each of the axes of x and y that are each uniformly distributed in their action upon the respective edges of a unit elementary square piece of a thin plate thus tending to bend it about both of these axes at once. Then there is a total resultant bending moment acting on each diagonal which is found by multiplying that on one side by $\sqrt{2}$. But this moment is distributed along a diagonal whose length is $\sqrt{2}$ times one side. Hence the intensity of the moment per unit of length is the same along the diagonal as along a side, and it may similarly be proven that the intensity is the same on any vertical plane whatever.

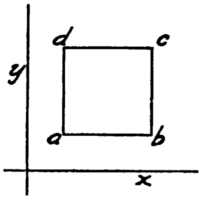


Fig. 14

When the equal moments about x and y are of opposite sign, the resultant moment would be of the same intensity on every plane but would not be normal to the surface on which it acts. It would then consist of a normal component which would be a bending moment, and a tangential component which would be a twisting moment. On planes at 45° with x and y the bending moment would vanish and the resultant moment would all be twisting.

External Moments. Moments of external loads and applied forces are designated as external moments. When a load upon a beam is multiplied by its distance from the support the moment thus obtained is that of the load about the support and by the principle of the lever when it is divided by the length of the beam the result is the magnitude of the reaction at the other support due to the load under consideration.

When the external bending moments are all about one and the same axis the resulting curvature is cylindrical and the bent surface is made up of parallel straight line elements. When the bending is in two directions and of opposite sign a saddle shaped surface is produced which is concave in one direction and convex in another at right angle to the first, and the applied moments involve twisting and warping as well as bending. The curvature of such a surface is called *anticlastic* as opposed to *synclastic* or dish shaped curvature such as a surface of revolution. Dish shaped curvature is produced by simultaneous bending in two directions when both curvatures are of the same sign.

Resisting Moment. The couple formed by the resisting stresses in the horizontal fibers of the material acting in tension and compression in a beam multiplied by a lever arm which is equal to the distance between the centroids of tension and compression respectively is the *internal bending moment* that opposes the external moment producing that bending. Likewise, where the external moment produces torsion or twist the internal moment opposing the external moment is the *internal twisting moment.*

Equilibrium of External Moments. I. The internal bending moment equals and balances the external moment of the applied forces where the member is bent in one direction only as occurs in cylindrical curvature, because there is no other kind of internal moment of resistance there present to produce equilibrium.

II. Where the moment of the external applied forces produces torsion or twist only as in a shaft, the external moment of the applied forces is balanced and equaled by the internal twisting moment of the shear couple since there are no other internal moments of resistance there present to produce equilibrium.

III. Where the member is bent in anticlastic curvature the external moments of the applied forces are balanced partly by the internal twisting moments and partly by internal bending moments because both are there present.

Deflections due to twisting or to shearing follow a fundamentally different law from those due to bending. For example, where twisting moments alone act as in a shaft, a simple summation of twisting deformations occurs along the axis of twist without increase in stress or twisting moment, which produces relative rotations whose amounts are equal to each other at the two ends of any portions of the shaft that have equal lengths. Likewise in a portion of a beam where bending moments of the same size act at all points there is a similar summation of shearing deformations horizontally along the length of the beam with equal increments of shearing deformation at equal distances from mid span, but the deflections follow an entirely different law.

4. Axial Forces. Let Fig. 15 represent a flat rubber band having a width *db*. Draw upon it a square *abcd* with the diagonal *ac* upon the axis of the band. Suppose the band is stretched lengthwise by the pull P. When the band is stretched in the direction of the pull it draws in laterally an amount *dd'* on one side and *bb'* on the other. It is stretched longitudinally the distance *cc'* so that the original square is elongated into a rhombus *ab'c'd'*. The material has undergone angular deformation *bab'*, *dad'*, similar to that under pure shear, as illustrated in Fig. 13, and shearing stresses are generated in the material along planes oblique to the axis of the direct stress.

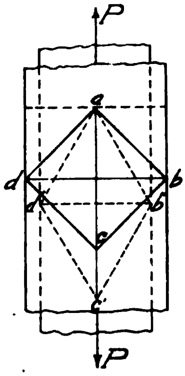


Fig. 15

The total axial force P which acts longitudinally across any right cross section *db'* of the band acts in the same axial direction across any oblique section making an angle v with the axis. It exerts a total normal component on this section of $P_1 = P \sin v$, and a tangential component $P_2 = P \cos v$. If the area of the right cross section is A, then that of the oblique section is $A/\sin v$. Hence the intensity of normal component is $P \sin^2 v/A$ and of the tangential component is $s_s = P \sin v \cos v/A$. The maximum value of s_s occurs when $v = 45°$ in which case $s_s = \frac{1}{2}P/A$ is one half the intensity on the right section.

This relation between shear and compression and tension explains in a measure the phenomena of rupture under compression, where in bodies having low shearing strength, rupture takes place by the material shearing off in planes more or less inclined to the action of the force, the inclination depending upon the degree of brittleness of the material. It further explains why on the surface of a polished flat bar of steel as the stress approaches the yield point value in tension, shear lines in relief crossing each other at 45 degrees appear and form a beautiful pattern standing out on the polished surface as shown in Fig. 16.

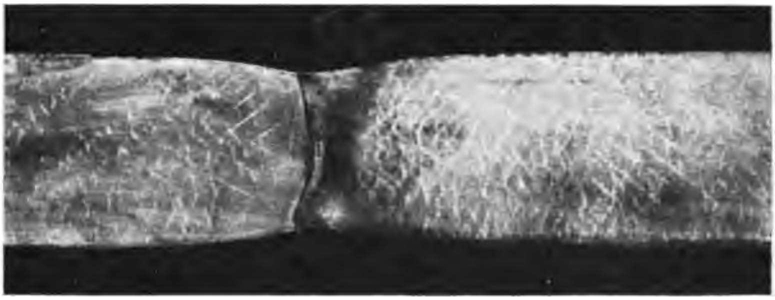


Fig. 16
Showing Shear Lines Developed in Tensile Test of Steel Bar Beyond Yield Point.

Referring to Fig. 15, it will be noticed that the rubber band grew narrower with the application of the tension. In other words, a strain occurred normal to the axis of the piece. When we consider the applied force *P* from the standpoint of statics it has no component whatever normal to the line of pull, but nevertheless the elastic action of the material produces a component strain in that direction which must be taken into consideration in arriving at the true stress not only in the case of simple pull but also where different stresses act upon the material in different directions.

This divergence between the deportment of elastic materials and the principles of statics of abstract forces may be stated as follows:

Any axial strain which increases the length of the bar is accompanied by a normal strain reducing the area of cross section. Likewise axial strain produced by endwise thrust is accompanied by lateral expansion or increase in area of cross section. In concrete columns of the hooped type practical advantage of this relation is taken to reduce lateral expansion by the embedment of steel hoops, thereby greatly increasing the ultimate resistance of the column under compression.

Approximate analyses in which these normal strains produced by direct stress are disregarded give a fictitious or statical determination of stress and strain as opposed to true stress and strain. These relations will now be investigated.

5. Elasticity of Volume. It thus appears that even the most elementary state of stress is a matter of considerable complexity. Simple experiments with prisms of soft vulcanized rubber will serve to convey a clear conception of some of the various relations of stress and strain.

A prism of this kind may be readily stretched to four times its original length and still return to its former shape. The elastic strains involved are thus enormously greater than in materials of construction, so much so that their relations to each other became readily apparent instead of being too small to be seen by the naked eye.

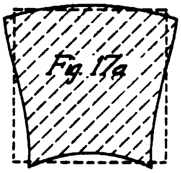
Under direct tensile or compressive stress the volume of the rubber prism remains practically constant. When elongated to four times its original length its cross sectional area is one fourth that which it had in its unstrained condition, and the breadth and depth of the prism are reduced to one half its original dimensions.

The ratio of the total lateral contraction to the total elongation in this case is one sixth, but this is not the true ratio for small step by step increments of stress. If we take for instance as the first increment an elongation of 1/10,000th of the length of a unit cube, it is obvious since the volume is constant that we might slice off a shaving on two adjacent sides 1/20,000th of a unit in thickness and add them both to the end, thus making up the 1/10,000th unit elongation with only the most negligible defect in total volume, so it is apparent that so long as the volume remains constant each infinitesimal increment of the elongation is accompanied by two lateral contractions each equal to practically one half that increment. Hence where opposing pressures act on four opposite sides of a cube and the volume remains constant the ratio of lateral contraction to longitudinal elongation, which we will designate as the coefficient K, equals one half, since if we slice off sheets of the thickness of the lateral compression from each of two adjacent faces and place them both on the end of the cube, the elongation on the end face equals the sum of both of the lateral shortenings or compressions on each face normal to the direction of the applied pressures.

When under direct stress in one direction only, the ratio of lateral contraction K, is less than one half the elongation or shortening the volume is increased under tension and decreased under compression.

If one vertical side of a rectangular prism of soft rubber, 1½″ square by 6 inches long, be ruled into 1/4″ squares, by longitudinal and transverse lines, and then bent in a vertical plane, see Fig. 17, the squares will be deformed into rhomboids whose sides are longer toward the convex face and shorter toward the concave face. The longer sides of the rhomboids nearer the surface convex outward are increased in length over that of the sides of the squares drawn on the face of the prism before bending. Likewise the lengths of that side of the rhomboids next to the concave face are shorter than before bending. The length of the squares along the center line of the prism has not changed from that before bending. In other words, there has been stretching of the outermost fibers in tension on the convex side, compression of the outermost fibers on the concave side, and no change whatever in length of the fibers in the plane called the **neutral plane or surface** lying mid way between these sides.

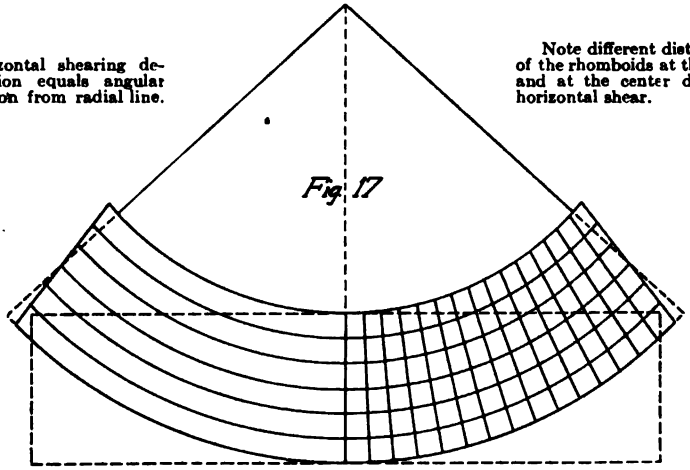
Figures 17 and 17a Showing Deformation of Rubber Prism Under Transverse Bending



CROSS SECTION AT THE CENTER

Showing transverse curvature, top and bottom of the beam, lateral contraction at the bottom and lateral expansion at the top.

Horizontal shearing deformation equals angular detrusion from radial line.



Note different distortion of the rhomboids at the end and at the center due to horizontal shear.

The concave and convex surfaces of the prism would correspond to the top and buttom surfaces, respectively of a beam under vertical load. But these longitudinal convexities and concavities of the bending produce transverse convexities and concavities. The concave upper surface Fig. 17 is compressed longitudinally. This crowding of the material together longitudinally, makes it bulge upward so that the transverse section of the upper surface is convex upward. Similarly the elongation along the lower surface draws the material in sidewise to replace material pulled endwise from near the middle of the bottom surface, and the bottom surface likewise becomes convex upward. The radii of curvature of each of the transverse sections for rubber equals twice the radius of curvature of the longitudinal section and for any other kind of material the ratio of the longtitudinal to the lateral radius of curvature equals K for that material.

Similar transverse deformations occur under load in a steel or concrete beam in a building except as modified by rigid attachments at the supports, and by the composite nature of reinforced concrete in the concrete beams to be discussed later. The transverse curvatures, however, are so slight that they are rarely noticed.

In a wide plate, when the ends of the plate are firmly clamped at the supports, the lateral contraction under flexure is much less than occurs in bending of a narrow plate because the natural curvature transverse to the axis of bending is resisted by the clamping effect at the ends, and the shearing rigidity of the material brought into play thereby opposes and reduces transverse deformation or may at times almost entirely eliminate it, a fact sometimes lost sight of in efforts to measure K experimentally.

Horizontal shearing distortions appear in bending the rubber prism Fig. 17 shown by the angular deformation of the rhomboids increasing from the center toward the end of the prism. These shearing distortions are quite distinct from the kind of stretching and compression of the material longitudinally occurring at mid length of the beam. The magnitude of this shearing distortion is measured by the angular deformation of the ends of the prism from the radius of curvature, it is an angular deformation outward from the end of the simply supported beam at the top or compression side and inward at the bottom or tension side. Such shearing distortions are termed **horizontal shears in horizontal planes** and are caused by the only kind of horizontal shearing forces present in purely beam action. In plate action, however, when the plate is bent in anticlastic curvature, that is, when it is so bent that the plate is convex along one horizontal axis and concave along another at right angles thereto, twisting occurs in the material about a horizontal axis, and we then have **horizontal shearing stresses in vertical planes** resisting such twisting, which will be discussed more at length later.

6. Pure Stresses. When the resultant of the forces acting on a body is normal to any given plane in it, it produces a pure normal stress on that plane. When the resultant is parallel to the plane it produces a pure shearing stress on that plane.

When a body is subjected to the action of forces in any direction the direction of the resisting stress acting upon any given interior plane may have any inclination to the normal and whenever it is not normal to the plane the stress will consist of both a normal and a shearing component.

Torsion of a circular shaft or prism about its axis develops shearing stress in all right sections of the shaft at every point of it in a direction perpendicular to the radius of the cross section that passes through that point. Every right section may be taken to remain plane, altho in fact the end surfaces bulge slightly at and near the axis, and other right sections near the ends bulge also but to a lesser extent. If parallel lines be drawn along the cylindrical surface of the prism any twisting will deform them into spiral lines on the surface of the prism.

In twisting the cylinder each elementary longitudinal fiber of the outer surface when it is changed into a spiral is elongated, and is stretched thereby. Other fibers of the prism also undergo alterations of length which are the same in all fibers at the same distance from the axis. This produces longitudinal shearing distortion along each cylindrical surface, which distortion increases in amount from the right mid section to each end of the cylinder and results in the bulging already mentioned.

Because of the symmetry of the cylindrical prism the contour lines of the curved end surface representing points whose deformation is of equal amount when measured from a plane normal to the axis will be concentric circles about the center. Because the spiral twisting causes compression toward the center, this has a tendency to produce elongation of the material along and near the axis while the longitudinal shear previously mentioned has a tendency to shorten the same material.

The end of a square prism under torsion changes the curvature of its surface in a still more complicated manner.

Suppose the torsion to be acting in a counter clockwise direction in Fig. 18, *xx* and *yy* being normal to the axis of the prism. The sides curve inward. The ends bulge in such wise that the length of the prism at the center is greater than at the surface of the sides. Moreover, the end is warped in such a way that in the octant *acy* it is concave, in *ycb* convex, in *bcx* again concave, and in *bcy* convex. In other words, in diametrally opposite octants the surfaces have the same curvature, either convex or concave, dependent on the direction of the twist. If the twisting torsional force acts counter clockwise when looking toward an end surface the octants behind the diagonals are convex and the octants ahead of the diagonals are concave and the contour lines of the octants are shown in Fig. 18. In a cylinder under torsion the contours would, however, be a series of concentric circles as shown in Fig. 19.

Figures Showing The Contours of the End Surfaces of Square and Circular Prisms Under Pure Torsion.

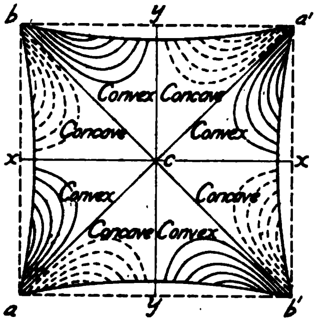


Fig. 18

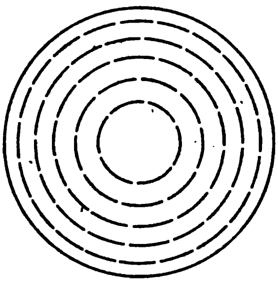


Fig. 19

Square figures or divisions which are drawn on the unstrained cylindrical surface of the India rubber prism upon twisting are deformed into rhomboids in which one diagonal is lengthened and the other shortened. This is typical shearing deformation. If the diagonals of these squares be also drawn in their unstrained condition the diagonals would present spirals of equal inclination around the unstrained cylinder but when strained one set of the spirals in the direction of the twist would be lengthened while those drawn in the direction opposite to the twist would be shortened.

In case of a square prism under torsion in which the sides are likewise divided into squares by lines lengthwise and crosswise presents some modification of the foregoing phenomena is brought about by the unsymmetrical distribution of the material about the axis of the prism. Upon twisting the square prism a similar deformation of the squares on the side surfaces occurs but the cross section, unlike that of the circular prism, is changed in form, owing to the twist in the side surfaces the sides of the squares no longer remain straight, because the sides of the initially square cross section of the prism become concave as seen in Fig. 18. Further, it will be noticed that the squares which were drawn on the unstrained prism are distorted to the greater extent at the center line of the side and least at the outer edge of the side. Therefore, the greatest angular or shearing distortion occurs under twist at points on the surface of a body nearest the axis and the least deformation occurs on the surface upon elements furthest away from the axis, a relation which is very different from that produced by transverse stress or bending. For in the latter, the intensity of the stress that causes failure increases to a maximum in the fibers furthest away from the neutral plane.

The distribution of strains brought about by twisting a prism about its axis, involving as it does shearing deformations that in any right cross section increase along each radial line with the distance from the axis and in each longitudinal fiber increase with the distance from the mid section has practically nothing in common with the distribution of strains which occurs in bending a prism about an axis at right angles to its length, altho when this bending is produced by a transverse load the longitudinal shears increase from the mid section to the ends the transverse shears as well as the longitudinal are greatest at the neutral axis, they are of the same sign both above and below the neutral axis and so produce bending deformation which is entirely different from twisting deformation.

Any attempt, therefore, to confuse the radically different effects of twisting and bending moments with each other is a manifest error. External or applied moments must consequently be carefully separated into their twisting and bending components in order to determine the kind of internal resistance that in fact hold them in equilibrium.

7. Equilibrium of an Element Under Shear. Because action and reaction are equal and opposite every tangential shearing force on the face of an element requires an equal and opposite shearing force on the opposite face. Two such shearing forces produce a rotary couple which must in some way be held in equilibrium or balanced by distributed or concentrated forces acting on opposite faces of the element or elsewhere. In Pure Shear the balancing forces are equal and opposite shears on the faces at right angles to the first pair of shears, and such distribution of stress may occur under flexure where a change of slope results from the rhomboidal deformation of the square elements and in pure torsion where such deformation occurs in the sides of the prism as before noted. But at those points along the length of a member under flexure where external restraint prevents rotary change in slope or horizontal shearing distortion, a different distribution of internal stresses is brought about by which equilibrium is maintained. For example, the supports of continuous beams built into columns are points where the tangent to the elastic curve is horizontal, and since rotary change in slope cannot occur, the balancing distribution of tensions and compressions at these points is brought about by external restraint. Such distribution would prevent rotation of *ad* about *d* Fig. 20 under the opposing shears *F* and *F*, shown in the figure. Uniformly varying stresses like those shown in Fig. 20 due to the rapidly varying bending moment at the supports will furnish the couple required to preserve equilibrium in this case. In fact any couple due to normal or tangential stresses whose moment is equal and opposite to that of the initial shears would maintain equilibrium.

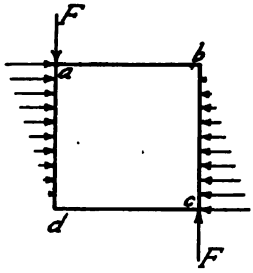


Fig. 20

Restrained Shear Stress. Where external restraint is present in addition to vertical shear on vertical planes in flexure, such shearing stress is sometimes called punching shear, notwithstanding the dissimilarity of the two actions. A better term for this particular kind of shear would be restrained shearing stress. The material under such restrained shearing stress offers greater resistance, for the reason that no horizontal shearing deformation occurs but only vertical shearing deformation. Hence the total deformation from shearing stress is only half as great as where by reason of flexure a rotary change in slope of the member also occurs.

8. **The Modulus of Rigidity** or shearing modulus is the ratio of the intensity of a right shearing stress to the amount of shearing strain produced by that stress. It is denoted by the letter *F*. Where the shearing strain is small it is equal to the tangent of the angle of

deformation and for a shearing stress of intensity q pounds per square inch we then have

$$\text{shear stress} = \text{shear strain} \times F, \text{ or}$$
$$q = \text{tang } \phi \times F = \phi \text{ (in radians)} \times F$$
$$F \text{ (pounds per square inch)} = \frac{q}{\phi} = \frac{\text{shear stress in lbs. per sq. in.}}{\text{shear strain in radians}}$$

Let *abcd* be the square face of a unit cube of material which is firmly fixed at *c* and *d*, and has a horizontal tangential or shearing stress of intensity q applied to its upper surface *ab*. This will bring into play an equal and opposite shearing resistance along *cd*, as well as equal and opposite vertical reactions at *c* and *d* which are equivalent to shearing stresses of intensity q on the faces *bc* and *da*. These all together deform the square *abcd* into the rhombus *ab'c'd'* Fig. 21 in which the angular deformation

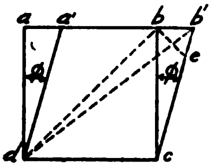


Fig. 21

$$\phi = \frac{aa'}{da} = \frac{b'}{bc} = \frac{q}{F} \text{ as above.}$$

The elongation of the diagonal *db'* is *eb'* cos 45° and the linear strain along each diagonal is

$$\frac{eb' \cos 45°}{db} = \frac{\frac{1}{2}bb'\sqrt{2}}{cb\sqrt{2}} = \frac{bb'}{2cb} = \tfrac{1}{2}\phi = \frac{q}{2F}$$

But this linear strain along the diagonal *db* is one half the shear strain just found for the side *ab*. This right shearing stress is eqiuvalent to an axial tension along *db* and an equal axial compression along *ac*. These together may be regarded as producing the angular deformation ϕ of which the tension produces one half as shown in Fig. 22 and similarly the compression produces the other half. Morever, the tension may be regarded as producing one half of the intensities of all of the shears as in Fig. 22 and the compression the other half. Fig. 21 may be made comparable with Fig. 22 by rotating *a'b'c'd* about *d* thru the angle $c'dc = \frac{1}{2}\phi$ as shown in Fig. 22.

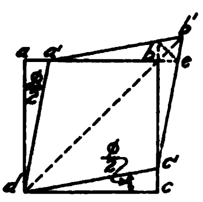


Fig. 22

9. Modulus of Elasticity. The normal unit stress or force in pounds per square inch, divided by the unit strain in inches, is the modulus of the material for that kind of deformation in pound inches. This coefficient is sometimes called Young's modulus, or the coefficient of elasticity.

To analyze a composite structure of concrete and metal upon elastic theory it is necessary to know the relative stresses under like deformations. Where these stresses are acting in parallel relation within the elastic limits of the material they bear the proportion of the ratio of the modulii of elasticity of the two respective materials for the intensity of strain involved. For steel the modulus E_s, is substantially constant for stresses well within the yield point value. Concrete on the other hand has no definite elastic limit; its modulus, E_c, decreases as the load increases, but tends to become constant under repeated stresses not exceeding one third to a fourth of its ultimate compressive strength. This fact must be kept clearly in mind in attempting to check mathematical analysis by practical measurements of deformation.

Both E_s and E_c remain substantially unchanged in changing from direct tension to compression. For steel, $E_s = 3 \times 10^7$ and is practically the same for structural grade and hard grade steel used for reinforcing when the stresses do not exceed two thirds of the yield point value, or 24,000 pounds per square inch for structural grade, and 34,000 pounds per square inch for hard grade, (see specifications).

E_c, the modulus of elasticity of concrete has a wide range depending on the aggregate used, the age, the range of stresses between which it is considered, and other conditions. The modulus for computation of the position of the neutral axes and for the resisting moment of beams is taken as:

(a) One fifteenth of that of steel, when the strength of the concrete is taken as 2200 lbs. per sq. in. or less.

(b) One-twelfth of that of steel, when the strength of the concrete is taken as greater than 2200 lbs. per sq. in and less than 2900 lbs. per sq. in., and

(c) One-tenth of that of steel, when the strength of the concrete is taken as greater than 2900 lbs. per sq. in.

10. Solid Hydraulic Stress. When an elementary cube or prism is subjected to stress of the same amount in three directions at right angles to each other, it is said to be subjected to solid hydraulic stress because the intensity of the stress is then the same in all directions.

Considère, in a series of tests on the compressive strength of concrete cylinders, found that the axial or endwise compressive resistance was increased by the application of sufficient lateral hydraulic pressure. The tests were carried to the extent of increasing the crushing resistance of the cylinders four to five fold. The increase in strength appeared limited only by the amount of hydraulic pressure applied laterally.

Plane Hydraulic Stress is the term applied in the discussion to follow to the condition of equal stress of the same kind in two directions within the material on planes normal to each other.

Pass any plane whatever through the line of intersection of the planes at right angles.to each other on which there are equal normal stresses then the stress upon such a plane is normal thereto, and the intensity thereon is the same for every such plane as for the normal planes. For consider a rectangular prism whose length is one unit perpendicular to the paper, whose height and width are a and b respectively, subjected to a uniform normal stress of intensity S on a and b, then the resultant of the horizontal forces Sa and the vertical forces Sb will be $S\sqrt{a^2+b^2}=Sc$, acting on the diagonal c, in a direction perpendicular to c, and with an intensity $=S$ per unit of c.

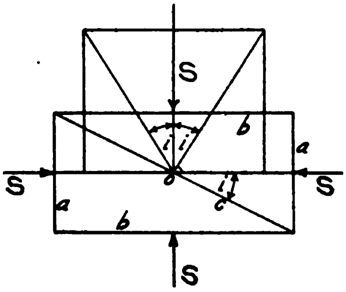


Fig. 23

The stress is evidently the same in all directions because we have assumed no fixed ratio between the length a and b and whether the prism be taken as wider or narrower the pressure on the surface c remains at S per unit of area, just as on a and b. Therefore, it is the same in all directions and is a true hydraulic stress in a plane.

Under plane hydraulic stress radial deformations and strains are equal in all directions, and circumferential stresses and strains likewise. In bending of plates this condition becomes a major element in accounting for their remarkable carrying capacity. The ultimate resistance of material under plane hydraulic stress is much greater than under normal stress in one direction only.

11. Right Shearing Stress. In Fig. 23, representing plane hydraulic stress due to equal normal stresses of the same kind on planes at right angles to each other, suppose that one of the equal stresses be made of opposite sign from the other. Then one of the components, say the horizontal component, is of opposite sign from that in hydraulic stress. The total resultant force, acting on the diagonal c remains unchanged in amount and intensity but no longer acts normal to c, but has normal and tangential components whose amounts depend upon the direction of c, Let c be inclined at an angle i to the horizontal then the resultant stress makes an angle $2i$ with the stress normal to c. When $a=b$ the cross section of the prism is square so that $i=45°$ and the plane of c makes an angle of $\pm 45°$ with the normal stress, the normal component in c vanishes, and the stress on both diagonals is entirely tangential. This is called a right shearing stress. (N. B. Every possible state of plane stress may be resolved into a hydraulic stress of a certain intensity superposed on a right shearing stress of suitable intensity). A state of right shearing stress at a point may therefore be defined either as consisting of a combination of two shears of equal and opposite intensity on planes at right angles to each other, or as a combination of a tension and a compression of this same intensity on each of two planes at right angles to each other making angles of $\pm 45°$ with the two shear planes first mentioned.

12. Effect of Like and Unlike Stresses, Acting at Right Angles. The ultimate resistance of material under plane hydraulic stress is greater than under direct stress of the same magnitude acting in one direction only. Hydraulic stress is brought about by stress of equal intensity of the same kind acting on planes at right angles to each other. Each stress produces an axial or normal strain in the direction of its application and a lateral strain of opposite kind normal thereto. Consequently, two normal stresses acting on planes at right angles to each other produce strains, which, since they tend to off-set each other do actually each diminish the other in the ratio of the axial strain to the lateral strain characteristic of the material under consideration.

But on the contrary, where two stresses of opposite sign act on planes normal (at 90 degrees) to each other, their coaction serves to increase the strains which either acting separately would produce. See Art. 2 and 4.

Because of this increase of the true strains over those computed to occur when normal strains are disregarded, safety requires the precise investigation of the amount of the increase or decrease of these stresses by lateral deformation in case of stresses of opposite sign on normal planes. Further, precise translation of deformation into true stress is requisite if elastic theory is to be checked by the measured elastic deflection of the structure under load, or if computed applied stress is to be compared with extensometer measurement of true strain. Without such precise translation, analysis by the elastic theory remains unsubstantiated, and its accuracy or inaccuracy is a matter of mere opinion rather than a matter of fact to be determined beyond dispute by scientific test.

13. On The Determination of True Stresses From The Measured Strains by Help of Elastic Coefficients

The modulus of elasticity E as just defined is the longitudinal unit stress p acting along x say, divided by the corresponding longitudinal unit strain e, in the same direction, viz. $E=p/e$, on the express condition that there is no lateral applied stress perpendicular thereto along either y or z. Hence longitudinal stresses can be correctly calculated as the product of the observed longitudinal strain e and the modulus E only in case there are no lateral forces acting along either y or z.

In this case of simple longitudinal stress there is, however, a lateral deformation or unit strain amounting to $—Ke$ along both y and z, and the apparent or applied stress p acting along x produces not only the longitudinal strain e along x but also two lateral strains each equal to $—Ke$ along y and z respectively.

In order to find out how much longitudinal strain the stress p would produce in its own direction x were it not also employed in producing lateral strain along y, let us superpose on the body a lateral stress of intensity $+Kp$ along y while the stress along z remains unchanged at zero. This applied stress of $+Kp$ along y when acting alone produces a lateral strain $+Ke$ along y and $—K^2e$ along z, but when acting in conjunction with the existing lateral strain $—Ke$ makes the resultant deformation along y vanish, while the total strain along z will be due to the stress p along z and to Kp along y, and hence will amount to $—K(1+K)p$ along z.

This deformation $+Ke$ along y which has been superposed on the body is necessarily accompanied by a superposed strain of $—K^2e$ along both x and z.

Hence the resultant strain along x due to the longitudinal stress p when lateral strain along y is entirely prevented and no stress is applied along z, amounts to $(1—K^2)e$, and the true stress along x as indicated by extensometer measurements of the strain will be $p_1=(1—K^2)p=(1—K^2)Ee$, which is the intensity of that part of the total applied stress p as calculated by help of E which produces strain in its own direction x. The rest of the applied stress along x, viz. $+K^2p$ is held in equilibrium by resistance to lateral flow represented by the lateral stress $—Kp$.

The foregoing calculation of true stress along x applies to all cases where the intensity of the lateral stress actually applied along y is such as to prevent deformation along y, while that along z is negligible, under which conditions the true stress tending to deform the body in the direction of x does not exceed $(1—K^2)p$ and the strain in that direction does not exceed $(1—K^2)p/E$.

These conditions are approximately fulfilled in a homogeneous flat plate or slab with multiple column supports.

Again, take the case of an applied longitudinal stress of intensity p along x and two applied lateral stresses along y and z of intensity q respectively. The resultant deformations will be those due to superposition of the effects of the separate stresses, viz.

Stress	Strain Along		
	x	y	z
p along x	p/E	$—Kp/E$	$—Kp/e$
q along y	$—Kq/E$	q/E	$—Kq/E$
q along z	$—Kq/E$	$—Kq/E$	q/E
Total strain	$(p—2Kq)E$	$(q–K(p+q))/E$	$(q–K(p+q))/E$
True stress	$p—2Kq$	$q—K(p+q)$	$q—K(p+q)$

The foregoing may be applied to the case of hooped columns, whenever it may have been determined either what is the absolute magnitude of the hydraulic stress q in the plane $y\ z$, or what fraction q the lateral stress may be of the longitudinal stress p.

The foregoing may also be applied to the case where q is of opposite sign from p by changing the sign of K, in which case the stresses and strains are increased instead of being diminished as they are when p and q have like signs.

14. Relations Between Elastic Constants. The foregoing investigations give rise to certain relations between E, the modulus of direct longitudinal strain, F the modulus of shearing strain, and K the coefficient for lateral deformation under direct stress. As already shown, the strain along the diagonal of a cubical block of material subjected to a state of right shearing stress of intensity q on two planes at right angles is $\frac{1}{2}\,\frac{q}{F}$, which may be replaced by $\frac{1}{2}\,\frac{p}{F}$ where p is the intensity of equal and opposite stresses normal to the diagonal planes, because the intensity is the same on all planes passing thru the line of intersection of the shear planes as already shown in Sec. 12.

The resulting direct stress p, in the direction of one diagonal only would, if acting alone, cause a strain $\frac{p}{E}$ in the direction of that diagonal, while the opposite kind of direct stress in the direction of the other diagonal perpendicular to the first, would, acting alone, cause a similar kind of strain to the above, amounting to $K \times \frac{p}{E}$ in the direction of the first mentioned diagonal. But each of these direct strains would be accompanied by a lateral strain of the opposite sign amounting to Kp/E. It is to be noticed that the two strains $\pm Kp/E$ at right angles to the plane of the right shearing stress annul each other. Hence the total strain along either diagonal is

$$\frac{1}{2}\frac{p}{F} = \frac{p}{E}(1+K)$$

from which $\dfrac{1}{2F} = \dfrac{1}{E}(1+K)$

$$1+K = \frac{1}{2}\frac{E}{F}$$

From this equation we may deduce approximate values of F when E and K are known:—

For Steel
$E_s = 3(10)^7$ and $K = .3$ Consequently, $F_s = 115(10)^5$

For Concrete—
When $E_c = 2(10)^6$ and $K = .15$ $F_c = 87(10)^4$
When $E_c = 2.5(10)^6$ and $K = .2$ $F_c = 104(10)^4$.

The value of K, when steel and concrete are working together in tension, is obviously neither that of the concrete nor that of the steel elements considered separately because the modulus of rigidity F of the mass has been increased by the embedment of the steel.

The characteristics of the coaction of the steel and the concrete in the composite mass which are analogous to the coefficient K for a homogeneous material will be discussed later.

15. Mechanical Work. In mechanics, work is the product of the magnitude of any force acting multiplied by the distance through which its point of application moves. In the mechanics of engineering materials the force applied per square inch of section performs work of deformation against the opposing elastic resistance or reaction of the material. This work is stored in the material which is in a state of stress by reason of this elastic resistance. On removal of the applied force the work is restored by the elastic reaction acting as a distributed force thru a distance that reduces the deformation to zero.

Internal Work of Deformation, therefore, is the product of stress multiplied by the distance through which it acts. If the stress increases from zero to the maximum amount which produces the unit strain, since the mean force is half the maximum, half the product of the increase of stress multipled by the unit strain, is the work of deformation per cubic unit of material. The total work of deformation is the sum total of the work done in deforming the several parts of the material.

It is to be observed that the term **work** has a very different significance in Mechanics from that of labor or exertion in the common usage of the word. A laborer for example, carries a pail of water horizontally to the men at work. In the horizontal movement of the pail which he carries he performs labor but does no mechanical work because the pail of water, while carried horizontally, does not gain or lose energy of position. This point may also be illustrated by considering a water tower. It requires mechanical work to pump the water to any height and when so pumped up it has energy of position and is capable of doing work as it runs down to a lower level.

Theory of Work in Elastic Structures. If a load be gradually applied to a floor made of elastic materials, the work performed by the load is half the product of its weight by the deflection or downward motion of the floor at the point at which the load is applied. The total external work of several loads on a floor at once is the sum of all the amounts of work performed by these loads separately. A beam, for example, transfers its load horizontally by help of the vertical shears set up in the material. Now so far as horizontal transference of the load is concerned no work whatever is performed, but the vertical shears produce bending moments with horizontal compressions and tensions which cause the beam to deflect downward under the load, and the load moves to a lower position or loses a portion of its potential energy or energy of position, and the amount of this last is the external work of the load as it is considered in structural mechanics.

According to the law of the concervation of energy this external work is equal to the internal work of deformation performed on the material of the floor by the tensions, compressions and shears called into play by the load.

The above relation was expressed in 1866 in the theorem of Clapeyron, (See Lamè, "Lecons sur la thèorie mathematique de èlasticitè des corps solides," deuxième èdition Paris, 1866) and is stated as follows:

> "The exterior force applied, multiplied by the displacement of its point of application in the direction of the force, equals the sum of all the internal work of a body elastically deformed."

This law is expressed in Merriman's Civil Engineer's Pocket Book, in the following statement:

> "If the system of bodies neither receives nor gives out energy, then its total store of energy, all forms included, remains constant. There may be a transfer of energy from one part of the system to the other, but the total gain or loss in one part is exactly equivalent to the loss or gain in the remainder."

This theorem is a direct corollary of the fundamental law of conservation of energy.

The law of conservation of energy may be stated as follows: Energy may be transformed or changed in form but it cannot be destroyed.

Applied to the floor it expresses the relation that the loss in energy of position of the load is transformed by reason of elastic deflection and resistance thereto into internal work of deformation. This principle is worked out in great detail in designing bridge structures to determine the deflections by work done, and is of the utmost value to the engineer in its various applications, some of which will be applied to concrete floors of Classes III and IV.

Resilience is the amount of work which may be stored up in a body under stress within the elastic limit and it will be given out again when the force producing the stress is removed. When the force has been applied so gradually that no energy is converted into heat then according to the law of conservation of energy the resilience must equal the external work. Resilience may also be defined as the measure of the ability of elastic material to restore external work.

Where the structure is not perfectly elastic some of the energy imparted to it by the load in its downward motion during deflection is transformed into heat and becomes dissipated. On removal of the load the recovery is either incomplete or the member returns to its original form slowly without ability to give back the full amount of energy expended in deforming it. In practical construction a floor of reinforced concrete exhibits nearly perfect elasticity after it has been repeatedly loaded. Initial loading, however, is certain to develop noticeable deviations from perfect elasticity because of shrinkage stresses developed in the hardening of the cast concrete. A few repetitions of loading largely eliminate such deviations, so that the theory of work and the elastic theory of deformations becomes applicable and proves a practical and precise method of treatment.

Dynamic Effect. When a load is applied instantaneously but without shock, instead of gradually, the work of the load is double that which it would have been had the same load been applied gradually. The effect of such instantaneous application accordingly, is to produce deformation or elastic strain of double the amount that would be produced by the same load applied gradually, and this is followed by an elastic reaction greater than the load and a vibration of the material up and down until it comes to rest finally at half of the maximum deformation or deflection. This fact forms the logical basis of the theory of impact which makes allowance for moving loads on bridge floors and the like.

16. Determinate and Indeterminate Structures. A statically indeterminate structure may be defined as one in which the elastic properties of the materials must be brought into the equations of equilibrium in order to determine the reactions, and from the reactions so determined the amount of stress in the various parts or members.

A statically indeterminate structure thus differs from a statically determinate structure; for the latter the reactions at the supports and the stresses are determined by the law of gravitation and the simple principle of the lever. For example, a beam which has only two supports and is freely supported thereon, whether it over-hangs those supports or only reaches from support to support, is statically determinate because the reactions at the supports may be determined from the moments of the loads and length of span on the principle of the simple lever. Where, however, such a beam has three or more supports, or two supports and restraint at these supports, so that for example the tangent to the elastic curve is horizontal at either of the points of restraint the elasticity of the material will in general have to be taken into consideration in order to determine the reactions where the loads are unsymmetrical.

There are two fundamental propositions which aid in the analysis of indeterminate structures. These propositions are known as the law of least work and the principle of rigidities.

17. The Principle of Rigidities and the Law of Least Work. In any load-carrying structure there are two principal kinds of stresses to be resisted by the structure — vertical shears and horizontal tensions and compressions that hold the applied moments in equilibrium. The distribution of stresses in indeterminate structures due to the shears and to the applied bending moments, although they cannot be determined by any statical considerations dependent upon the law of gravitation and the principle of the lever, are however, fixed by another law of nature just as invariable in character as either of these, namely by one which takes account of the relative stiffness of the several parts of the structure. This law is the law of rigidities which causes the load in its travel to the support to divide itself between the several possible paths into parts strictly in proportion to the rigidity of those paths. This principle, therefore, determines the predominant action of the structure and is susceptible of a very simple illustration, the application of which establishes certain useful relations that can be conclusively demonstrated. To illustrate this point, let a load be hung up to two springs of equal length side by side, one of which is twice as stiff or rigid as the other.

If W is the amount of the whole load, the load on the weakest spring is $\frac{W}{3}$ and that on the other spring would be $\frac{2W}{3}$. Under this arrangement the more rigid spring will carry the larger portion of the load, two pounds to each pound carried by the other. So in any structure where there are several parts of the structure, each of which is to carry some part of the load, the question of how the load is to be divided between them is determined by their relative rigidity or stiffness. That part which lacks stiffness and yields easily will carry little of the load and will leave the brunt of the work to those parts that are more rigid and unyielding.

Now the law of least work states that in any indeterminate elastic structure, *i. e.* one in which the internal stresses are detremined by the law of rigidities, the sum total of the internal work stored in it is less than would be stored by the same applied forces in case the internal stresses were distributed according to some other law than that of rigidities.

Since the case just considered is one where the rigidities determine the stresses, consider this a case where the law of least work holds true. If we designate the deflection as D and the stiffness of the weaker spring as e, that of the other will be $2e$. Then $D = W/3e$ and the work expended on the weaker spring when the load is applied gradually is the product of the deflection D by the mean force applied which is one half the final load $W/3$, or $DW/6 = W^2/18e$.

Similarly the work expended on the other spring is $DW/3 = W^2/9e$ and the total work expended on both together is $\frac{1}{2}WD = W^2/6e$. Now if we suppose the weight W to be in the form of a long horizontal rod and that it is suspended by the two springs one situated at each end of the rod, then the system is statically determinate and each spring necessarily carries a load of $W/2$ so that the total weight W is not divided between the springs in accordance with the principle of rigidities but instead by the principle of statics. The principle of least work would require that with this new arrangement more work should be done upon the springs than in the previous arrangement. That this is in fact the case may be readily shown by computation as follows:

The deflection of the weaker spring is $D_1 = \frac{1}{2}W/e$ while the work of the weaker spring is $\frac{1}{4}WD_1 = W^2/8e$. The deflection of the stronger spring is $D_2 = \frac{1}{2}W/2e$ while the work of the stronger spring is $\frac{1}{4}WD_2 = W^2/16e$. Total work $3W^2/16e$, which is larger than before, as it should be according to the principle of least work.

Any other arrangement except such a one as divides the load in proportion to the rigidities will likewise make the expenditure of work exceed that first found. For example, let the weaker spring carry the entire weight W by supporting it at its center of gravity while the other spring is attached to the end of the rod and carries nothing. Then the deflection of the spring carrying the load is W/e and the total work $W^2/2e$, a result also larger than that first obtained. Again, if the stiffer spring is used to support the total load and the weaker none then the deflection would be $W/2e$ and the total work $W^2/4e$, also larger than the first result.

It thus appears that the relative magnitude of the results is independent of the actual stiffness e and $2e$. All that affects the relative amount of work done is the relative rigidity of the springs, i.e. the fact that in the case proposed above one spring is twice as stiff as the other. This law of rigidities is to be applied not only in determining the shears or reactions at the supports and the distribution of internal stress in a slab, but also the distribution of moments between the several columns and floor panels under unbalanced loading.

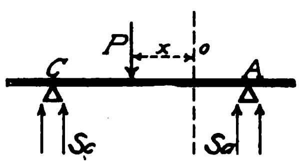


Fig. 24

The relation of shear and moments as dependent upon the rigidity of the restraint at the supports may be expressed in a simple manner by considering a section of a continuous or restrained beam of length $CA = L$, having a moment M_c and a shearing reaction S_c just at the right of the support C. Then if the bending moment at any section O be M and the shear S, we have, if x be measured from O,

$$M=M_c+\left(S_c-\sum_{c}^{o}P\right)x$$

$$\frac{dM}{dx}=S_c-\sum_{c}^{o}P=S$$

Move the original O to A and designate the moment at A as M_a and distances from A by a, i. e. put $x=a$, then

$$S_c=\frac{1}{L}\left(M_a-M_c+\sum_{c}^{A}(Pa)\right)$$

Therefore S_c the shear at the support C, is the algebraic sum of the restraining moments at the supports, plus the moments of the loads about the other support divided by the span length. Also because the moments at the supports depend upon the relative rigidities at the supports the reaction S_c and so the shear S at any point O as also the moment M at that point is determined by the relative rigidities. Moreover, since $\frac{dM}{dx}=S$, we see that at points *where S passes through zero, M has a greater numerical value than at any nearby points.*

18. Predominant Action: Applied mechanics differ from theoretical mechanics since no mathematical theory can take into consideration all of the subsidiary actions which are involved in a practical structure. It is necessary to determine its predominant action or mode of operation and the precise mathematical stresses are then calculated for that mode of operation under the applied forces and the determinations so made are applied in practice with such modifications as subsidiary actions may demand.

For example, consider the stringers of a railroad bridge: They are connected to the floor beams by angle iron connections which have a certain amount of rigidity but the restraint imposed by the line of rivets connecting the stringer to the beam is wholly insufficient to cause the stringer to act as a continuous beam and it is treated as though its action were solely that of a simply supported beam, the subsidiary effect of the small increase in strength due to the added rigidity of the connection being wholly disregarded.

In a similar way, it is necessary to analyze any combination of reinforced concrete and metal. Consider for example, a slab of concrete of uniform thickness supported on the tops of parallel spaced beams with one percent of reinforcement in the bottom throughout. Were the material homogeneous, such a slab would be computed on the continuous beam theory but because of the lack of rigidity of the concrete in tension over the supports compared with the rigidity of the central section between the supports, reinforced and made rigid by embedded steel, the relative rigidities fix the predominant action as that of a simply supported beam. The line of inflection moves toward the support as the relative rigidity of the central section increases over the rigidity of the section at the support until all cantilever action becomes negligible and the operation of such a slab is properly treated as a flat arch or simple beam.

In the case of a continuous line of interior beams rigidly built in between columns and reinforced for continuous beam action, there are present two separate and distinct kinds of action, beam action proper and arch action. In the beam action, horizontal tensions in the steel are balanced by horizontal compressions in the concrete on the opposite side of the neutral plane. Arch action is brought about by the rigidity of the support which enables it to take horizontal thrust at the bottom balanced by horizontal compression or thrust at the crown or central top portion of the beam. The relative rigidity of the *arch action increases with the relative increase in depth of the beam and decreases rapidly with the decrease in relative depth of the beam or slab, until where the thickness is $L/20$ or less, arch action is practically negilgible. As the percentage of steel increases the amount of arch action decreases because the tension zones are rendered relatively more rigid by the increase in reinforcement.

In an end panel where the wall columns are long and light, flexure of the columns will destroy the lateral rigidity of the support as an abutment and practically eliminate such action. So that in an end panel the beam normal to the wall column can be figured from the standpoint of beam action only and the degree of restraint at the support which determines whether the beam is to be treated as a restrained beam or a beam partially restrained at the end would again be determined by the relative rigidity of the wall column and the interior column by the application of the law of rigidities.

The coaction of the metal and concrete and the internal stresses of bond shear are to be investigated further by this same law as also are the various paths through a slab by which the load may travel to each support when these supports exceed two in number.

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*Chapter II, page 51, note test quoted showing percentage of load moment and steel stresses in beams and girders from 64% to 80%. The difference between these percentages and 100, less the negligible effect of the direct tensile strength of the concrete represent arch action.

19. Bending in Multiple Directions. Consider a uniform horizontal flat plate in which stresses are produced by loading. At any point *P* let a vertical plane of section be regarded as dividing the plate into two parts. Then the entire action experienced by either part of the plate per unit of section by virture of the internal forces between it and the adjoining part may be called a stress in the plate. This stress at a point of a plate will consist so far as horizontal forces are concerned of tensions on one side of the neutral surface and compressions on the other side of it, which increase uniformly in intensity with their distance above and below that surface but these tensions and compressions may make any equal angles with the normal to the plane of section. As this plane of section is revolved about the vertical the horizontal stresses will vary in tensity and direction, and at a point situated at any assumed depth in the plate we shall have a state of horizontal plane stress which will be completely defined by knowing the direction and intensity of the stress on any two vertical sections. The only variation that will occur in the state of stress at points at different depths along a vertical line at *P* will be a variation in intensity from a largest positive value at one face of the plate to a numerically largest negative value at the other face. There will be no change of direction of the stress with the depth. Like all states of plane stress there will in general be only two planes of section on which the stresses will be normal and these planes will be at right angles to each other. On all other planes the stresses will not be normal but will have shearing components.

The normal components being of opposite signs above and below the neutral surface will constitute bending moments on that section or about that section while the shearing components being also of opposite signs above and below will constitute twisting moments about the normal to that section. Consequently the stress at any point *P* of a plane of section of a plate will be resolved into a normal component that gives rise to a bending moment about an axis parallel to the section and a tangential or shearing component that gives rise to a twisting component about an axis at right angles to the section. So that just as in simple stress there are but two vertical planes of section at right angles to each other on which the entire stress is a bending moment. These are called principal planes. All others have a twisting component about the normal to the section.

When the normal components on one side of the neutral surface are all of one sign and all are of the opposite sign on the other side, the surface becomes concave on the side where the compressions occur and convex where they are all tensions. The curvature is then said to be synclastic or dish shaped. But when the stresses have opposite signs on the two principal planes in each surface of the plate, the curvature is said to be anticlastic or saddle shaped, and two planes of section may then be found on which the entire action is a twisting moment, because in revolving the plane of section from the position of a principal plane where the total action is a bending moment that produces compressions at the top of the plate to the position of the other principal plane where the total action produces tension at the top there must be some intermediate position where the bending component vanishes.

Where the state of bending stress at a point *P* in a plate is such that the principal stresses have the same sign and are equal to each other then the intensity is the same on every vertical plane of section at *P* and a state of simple synclastic stress is said to exist at *P*. Simple synclastic stress produces spherical curvature in a uniformly flexible plate. When, however, the principal stresses are equal in amount but of opposite signs, the intensity of the stress is the same on every vertical plane of section, and the two planes on which the stress consists of twisting moment alone, make angles of 45° with the principal planes. Such a state of stress in a plate is said to be a simple anticlastic stress, and produces anticlastic flexure in a uniform plate such that the curve of section of the two principal planes are alike but lie in opposite directions.

Simple synclastic stress is analogous to plane hydraulic stress and simple anticlastic stress is analogous to plane right shearing stress. As previously stated, any given state of plane stress may be regarded as made up of a hydraulic stress and a right shearing stress of such intensities that superposed on each other they will reproduce the principal intensities of the given stress unchanged. In the same manner any given state of stress at a point *P* of a plate may be regarded as made up of a simple synclastic stress, and a simple anticlastic stress of such intensities that superposed on each other they will reproduce the resultant intensities on the principal planes of the given state of stress.

It follows that at any given point *P* of a uniform plate where the principal curvatures of the plate do not vanish and are unequal the only planes on which the twisting moments vanish are the principal planes at right angles to each other. The stress on every intermediate plane exerts a twisting moment about a normal to the section. Points of support, and panel centers are in general the only points where spherical curvature exists, and are consequently the only points where no twisting moments occur on any planes of section. Elsewhere twisting moments occur on all except two principal planes. For the analytical development of these and other principles covering the stresses in flat plates, see Thomson and Tait's Natural Philosophy.

It follows from these universally accepted principles that in flat plates and slabs in which double curvature exists, as it does when they rest on or are integral with separated supports or beams running from support to support, the essential phenomena to be looked for and considered in every vertical section are the intensity and distribution both of the bending moments and the twisting moments. The bending moments produce stresses normal to the sections. The applied twisting moments are resisted by horizontal shears in the section which are of opposite signs above and below the neutral surface, and are held in equilibrium either by shears of equal intensity in vertical planes at right angles, or by twisting moments on horizontal planes, as will be explained later.

Any attempt to apply beam strip theory which depends upon cylindrical curvature and bending moments alone to cases of synclastic curvature can only lead to a fog of uncertainty and to results necessarily incorrect because of its radical divergence from the immutable laws of statics as applied to states of stress.

Cases of synclastic or double curvature which are concave upward like those just mentioned are produced by loads upon the upper surface of a panel which has supports around it, but a plate supported at its center having symmetrical loads upon its upper surface will not necessarily undergo synclastic curvature and become convex upward. If its edges are free it is almost certain to bend in cylindrical curvature about whatever axis offers the least moment of resistance and bend practically none at all about an axis at right angles to this. But if its edges are restrained so as to remain practically in a single plane, as they are for example where the convex plate considered is either a portion or the whole of one panel of a continuous plate of several panels then this continuity furnishes the restraint required to ensure double curvature. The stresses required to so restrain the plate to a state of double curvature and prevent it from assuming single curvature are insignificant in amount and are practically negligible. A circular plate for example supported at its center and uniformly loaded, were its edges restrained so as to make it assume convex curvature with all of its meridian sections alike and then its edges were suddenly set free would be in a state of unstable equilibrium as to its curvatures which would not persist however, for the greatest principle curvature would immediately become cylindrical about its axis of least resistance, and all sections perpendicular to this axis would become parallel straight lines. All cantilever plates hereafter treated are subject to these conditions.

20. The Applied and Resisting Moments Parallel to the Sides of Uniformly Loaded Square Panels Supported on all Four Sides:

Let *AOBD* Fig. 25 be one quarter of a square panel of a flat plate of which *O* is the panel center and *D* one of the four corners where the ends of the stiff side beams that surround the panel are supported, so that *A* and *B* are the middle points respectively of the two side beams that end at *D*.

Let *W* be the total uniform panel load then the total uniform load on $AOBD = W/4$ and so far as statical moments are concerned it may be taken to act at the center of *AOBD*. This load has a statical moment of $WL/16$ about any axis parallel to either side when taken with the equal reaction of the corner support in case *L* is the length of the side, and each of these two independent applied moments at right angles to each other must be independently held in equilibrium by resisting moments in those portions of the beams and slab that are in this quarter panel.

In order to find out just what these resisting moments are the distribution of the load upon the beams as well as the shearing stresses in the panel must be considered. The load upon the beams is uniformly distributed, viz., $W/8$ upon *AD* and *BD* respectively, thus producing a bending moment of $WL/32$ in each side beam. Each beam consequently resists one half the total applied moment in that direction in *AOBD*.

Now consider the manner in which the quarter panel of the slab transfers the load to the beams by the shears and moments produced in the slab. It is evident by symmetry that *OA*, *OB* and *OD* are lines of zero vertical shear at all points. In fact, every vertical plane through *O* cuts the slab in a section of zero vertical shear at all points in case the side beams are uniformly loaded, as they must be provided the stiffness of the slab is small per unit of width compared with that of the beams.

Subdivide AD and BD each into the same number of equal segments and join these points of subdivision with O. Then the elementary triangles so formed are equal to each other in area, and if any line as PE be drawn parallel to AD the vertical shear in each of the segments of PE is equal to the load on the triangular area between it and the two sides radiating from O, because these sides being lines of zero shear, the whole load on the triangle must be supported by the shear on that segment of PE which constitutes the base of the triangle in which alone a vertical force is applied in the form of a shear. This subdivision into elementary triangles consequently transfers the total load $W/8$ which is situated upon the triangle AOD to the beam AD and distributes it there uniformly along AD and in so doing calls into play a moment $WL/48$ which is the product of the load $W/8$ by the distance $L/6$ between the center of gravity of this triangular load and the beam AD which supports it.

In order to consider more in detail this transfer of the load $W/8$ from the center of gravity G of AOD to C its center of support on AD along which $W/8$ is uniformly distributed, introduce two equal and opposite vertical shears $\pm W/8$ at K the one third point of AD. Then $AC=L/4$, $AK=L/6$, $KC=L/12$, and $GK=L/6$.

In transferring the load $W/8$ from G to AD, the resultant of all the vertical shears on planes like PE parallel to AD is a vertical shear $= \pm W/8$ at G, and the resultant of all the vertical supporting shears at the edge of the beam AD is a vertical shear $= -W/8$ at C. Now introduce at K the equal and opposite shears $= \pm W/8$ which are in equilibrium with each other and are parallel to the resultants at G and C. These two shears so introduced at K will not disturb the equilibrium already existing and may be considered in combination with the forces previously enumerated.

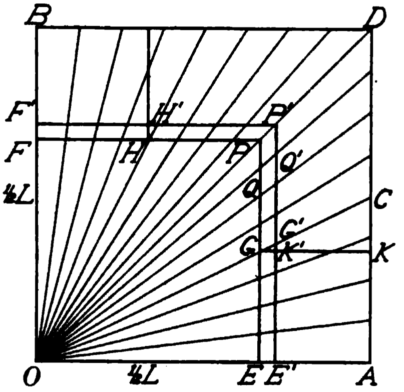

Fig. 25

Take the couple of shears $+W/8$ at G and $-W/8$ at K. They produce a moment $WL/48$ about any axis parallel to AD, which is a bending moment because the shears are in vertical planes parallel to the axis of moments. This is the bending moment in the slab which has been already obtained.

Again, take the couple of vertical shears $+W/8$ at K and $-W/8$ at C, a couple of shears both of which act in the vertical plane thru AD, and which therefore constitute a twisting moment about an axis perpendicular to AD of an amount equal to the product of $W/8$ by $L/12$ viz. $WL/96$. Hence the loads upon AOD and BOD together produce a bending moment of $WL/48$ and a twisting moment of $WL/96$ or a total of $WL/32$ about each of two axes parallel respectively to the sides AD and BD. These bending and twisting moments are resisted by the material of the slab, and serve to transfer the load to the beams so that of the total applied statical moments $WL/16$ about each side, one half is resisted by bending moments in the beams, one third by bending moments in the slab and one sixth by twisting moments in the slab.

Since every element of load on a plate produces some twisting moment it appears that twisting moment is an essential constituent of "plate action" as distinguished from "beam action".

The measure of the twisting moment in this combination of beams and slab is one sixth of the total statical applied moment of the load or one third of the total statical moment applied to the slab.

Since the secret of slab action resides in the twisting moments it is desirable to further consider the manner in which these moments arise: In order so to do, consider first the twisting of a bar or shaft of rectangular cross section subjected to a uniform twist from end to end by the application of equal and opposite twisting moments at its two ends respectively. It is known that the greatest shearing stresses in the right cross sections of such a shaft are found at the middle of the longer sides and that they act parallel to those sides, altho at first sight it would seem as if the greatest intensity of shear would occur at such points as are at the greatest radial distance from the axis of twisting. Such, however, is not the fact because it is the relative rigidities of the several parts of the rectangular cross section which determine the distribution of the resistance to twisting. It is evident that the total resistance to shearing distortion between two neighboring rectangular right sections of a wide bar.is much greater along the long side than across it, for any shearing distortion can be produced in the thickness of the bar more easily than in its breadth, whether it be simple shear or a twist about the axis. Hence it appears that the intensity of the distribution of the shears in the right section of a rectangular bar is really controlled by the relative shearing rigidity of its parts and not by the relative distance of these parts from the axis of twist as might at first be supposed. The relative intensity of these maximum shears on the long sides is dependent upon the relative length and breadth of the rectangular cross section, being larger the greater the relative length of the section compared with its breadth.

This admitted experimental and theoretical result is intimately related to Kirchhoff's theorem as to the conditions of equilibirum at the boundary of a plate subjected to external forces, which theorem states that when a twisting moment of given intensity per unit of width of section of a horizontal plate is produced by vertical shears in that section or bounding surface the same twisting moments may be otherwise produced by applying instead equal and opposite horizontal shears in the same bounding section, and any difference in effect between the action of twisting moments of equal intensity produced in these two different ways will not extend more than an infinitesimal distance from the section into the plate.

Now let a vertical plane $P'E'$ parallel to AD pass close to PE, then the magnitude of the total twisting moment acting between PE and $P'E'$ about AO is the product of the total vertical shear S across PE taken as acting at G muliplied by the distance $G'K'$ by which it is shifted along PE in being transferred to G'. Since the shift parallel to AD of the vertical shear on opposite faces of any one of the elementary parallelograms that make up the area $PEE'P'$ increases uniformly from E to P, the intensity of the twisting moment about AO applied to PE increases uniformly along EP from zero at E on the median line to P on the diagonal. In this reasoning the section PE which is employed has no reference to the center of gravity G, but is a section drawn parallel to AD anywhere between O and AD. Moreover the total twisting moment on PE increases uniformly with its distance from O, *i. e.* is proportional to the length of PE itself.

Now the application of the facts mentioned, respecting the twisting of rectangular shafts and Kirchhoff's theorem on boundary conditions, to flat plates or slabs where the twists are produced by the shifting laterally of vertical shears is this:—

In a vertical rectangular section of a plate in which the thickness is so small compared with the breadth as it is in a floor or slab all the shearing resistance to a twist in the section practically consists of horizontal shears of opposite sign above and below the neutral surface which increase from zero intensity at that surface to their greatest positive and negative values at the upper and lower faces of the slab, and this notwithstanding the fact that these twists are produced by the shifting laterally of vertical shears between this and parallel sections.

Having thus discovered the distribution of the twisting moments in their action upon any elementary prism of the strip *PEE'P'* whose horizontal cross section is *PQQ'P'* for example, which extends vertically thru the thickness of the plate or slab let us consider the action upon this prism of the applied bending moment and twisting moment somewhat more in detail.

Let $+H$ and $-H$ be the horizontal shears that act parallel to *AD* on the vertical edges of a horizontal lamina *PP'Q'Q* one unit thick. The intensity of these stresses $\pm H$ depends upon the distance of this lamina above or below the neutral surface; those above the neutral plane will tend to twist the prism in one direction, while those below it will tend to twist it in the opposite direction, the entire system of shears being held in equilibrium by the twisting resistance of the prism about its vertical axis. There are no vertical or horizontal shears upon the vertical faces *PP'* and *QQ'* since these lie in planes of zero shear, but there are opposite normal stresses $+T$ and $-T$ upon the faces *PQ* and *P'Q'* of different signs above and below the neutral axis which are due to the bending moment. Since $+T$ and $-T$ altho parallel are not along the same straight line, those acting on every lamina above the neutral plane will produce a twist about a vertical axis which will be held in equilibrium by an equal opposing twist due to the forces acting on corresponding lamina below the neutral plane. The above mentioned twists and any twists due to normal stresses on the vertical faces *PP'* and *QQ'* are resisted by twisting shears in the neutral plane and planes parallel to it, whose intensity decreases with the distance of the plane from the neutral surface and they call for no resistance elsewhere.

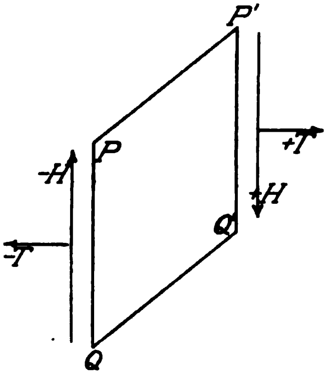

Fig. 26
Horisontal Lamina or element of vertical rhomboidal prism extending thru the plate.

From this explanation and exposè it is evident that the horizontal shears on the vertical sides of a prismatic element extending thru a plate have no necessary relation to the horizontal shears on vertical planes at right angles to them since they each merely produce independent self-contained twists in the prism, and the proposition which has been used in establishing the accepted fundamental differential equations of plates to the effect that the intensity of twisting moments is the same on any two vertical planes that cross at right angles is not necessarily true since it is based on incorrect assumptions which omit to consider twists about the vertical.

21. The Applied Bending and Twisting Moments About the Diameter of a Circular Cantilever Plate Restrained at its Center:

(*1*) *Uniform load thruout.* Let $AOBD$ Fig. 27 represent one quarter of a uniform circular plate of radius $= R$ restrained at O. Let W be the uniformly distributed load upon the entire plate, then the total uniform load upon $AOBD = W/4$, and so far as moments about AO or BO are concerned, this load may be taken to act at C the center of gravity of this quadrant at a distance $4R/3\pi = 0.424$ R from AO and BO. This load when taken with the equal reaction of the central support has a statical moment $= WR/3\pi$ both about AO and about BO independently.

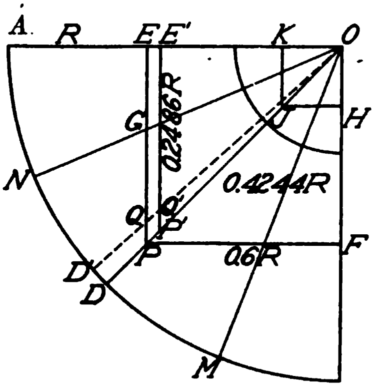

Fig. 27

Each of these applied moments must be held in equilibrium by resisting moments in the material of this quarter of the plate, and in order to determine in more detail how this is accomplished it may be noted that all diametral sections thru O cut out by radial planes are sections of zero shear at all points of these sections, and that the load is transferred by vertical shears to O along radial lines only. For example, the load upon any area $ADPE$ is carried by the vertical shear in the section PE between it and O. Now let D be the middle point of the quarter circumference AB, Then the load $W/8$ upon the octant ADO may be regarded as transferred to O by shear in sections of ADO parallel to BO and by the moments that these shears produce. Let G be the center of gravity of the load on the octant ADO, then $OE = 4\sqrt{2}R/3\pi = 0.6R$, and $GE = OE$ tan 22°30′ $= 0.25R$ nearly.

The resultant of all the shears in ADO may be taken, so far as the moments produced by them is concerned, as a resultant shear $W/8$ acting at G the center of gravity of ADO in the section PE

parallel to *BO*, while the reaction of the support which supports this part of the load acts either at *O* or near by at some section *JK* of the column cap parallel to *BO*. At *E* introduce two equal and opposite vertical shears, $\pm W/8$ in the plane *PE* which will in no way disturb the action of other shears. Then the vertical shear $-W/8$ at *O*, taken with $+W/8$ at *E* acting parallel to *BO* will produce a moment about *OB* of $OE \times W/8 = 0.6\ R\ W/8$, which is a bending moment because the shears producing it act on planes parallel to the axis of moments *OB*.

Again, the vertical shear $-W/8$ at *E* and $+W/8$ at *G* acting in the plane *PE* will produce a moment about *OA* of amount $GE \times W/8 = 0.25\ R\ W/8$, which is a twisting moment about *AO*, because the shears producing it both act in the plane *PE* at right angles to the axis of moments. These two moments and the shear $W/8$ at *G* parallel to *OB* transfer the load in this octant to *O*. Likewise the shear $W/8$ in *BDO* acting in vertical planes parallel to *AO* transfers the load on the octant *BDO* to *O* by help of bending moment $= 0.6\ R\ W/8$ about *AO*, and a twisting moment about *BO* $= 0.25\ RW/8$.

The total statical moment about *AO* or *BO* is the sum of the bending and twisting moments about either axis, viz.

$$(0.6 + 0.25)\mathrm{R}\ W/8 = 0.425\ R\ W/4$$

which is the same amount as was obtained when the quadrant was treated as a whole. But this detailed analysis has established the fact that the statical moment about each of the independent axes *AO* and *BO* at right angles to each other, viz. $0.424\ R\ W/4$, is made up of two parts, only one of which is a bending moment, viz. $0.3\ R\ W/4$, so that the bending moment is only 70% of the total statical moment, while the twisting moment which measures the slab action amounts to 30% of it, in case the reaction of the support is concentrated at the point *O*. But in case the reaction of the support is taken to be uniformly distributed over the area of a circular cap of radius $= R_0$ the bending moment about a diameter will be only $0.3(R - R_0)W/4$, so that when R_0 has a minimum value of $0.2R$ the bending moment is only $0.24\ R\ W/4$ or less than 51% of the statical applied moment.

The distribution of the twisting moment along any section *PE* depends both on the intensity of the vertical shear along *PE* and on the shift of shear between any segment *PQ* and the adjacent parallel segment *P'Q'*. This shift is zero at *E* and increases continuously from E to P because of the decreasing angle at which PE intersects the several radii between AO and AD. The twisting moment along EP reaches its maximum intensity at P on the radius that makes an angle of 45° with PE, and total twisting moment on any section parallel to PE becomes greater the greater its distance from A where it is zero. It is evident therefore that the intensity of the twisting moment along any section that is a chord of a circular cantilever plate, is zero at the middle of the chord, and increases along the chord to the point where it intersects a radius at 45° with it after which it diminishes to zero at the edge of the plate.

(*2*) *Uniform load around the circumference:* Let O be the center of the plate and ADB be one quarter of the loaded circumference which carries a load $W/4$, whose center of gravity is situated at a perpendicular distance of $2R/\pi = 0.6366R$ from either AO or BO, so that its statical moment about AO or BO is the product of $W/4$ by this distance, viz, $RW/2\pi = 0.6366RW/4$.

Now consider the separate loads $W/8$ upon the two octants AD and BD. The center of gravity of a uniform load on AD is situated at perpendicular distance of $2R\sqrt{2}/\pi = 0.9R$ from BO, and at a distance of $0.9R \tan 22°30' = 0.373R$ from AO so that the load $W/8$ upon AD has a moment $= 0.9RW/8$ about BO and a moment $= 0.373RW/8$ about AO.

The first of these moments is a bending moment and the other a twisting moment. Again, when we consider the load $W/8$ upon the area BD it produces a bending moment and a twisting moment about AO and BO respectively of the same magnitudes as those just computed; so that the total moment about either AO or $BO = (0.9 + 0.373)RW/8 = 0.636RW/4$, i. e. the same as that previously calculated. This twisting moment is more than 41% as large as the bending moment.

It is easy to find the position upon ON of the center of gravity of a composite load on AOD which is made up of one part uniformly distributed over the area of the plate and another part uniformly distributed around its circumference in case the relative magnitudes of these parts are known. It is evident that the magnitude of the twisting moments produced by such a composite load about any radius is more than 40% of the bending moment about that axis, since that is the fact respecting each of the component loads separately. The total statical moment about the axis is the sum of the twisting and bending moments taken together.

It will, however, conduce to simplicity in the calculation of the total shears and moments if they be calculated for each of these two kinds of distributed loads independently, when their combined effect will be simply the sum of their separate effects.

22. Work of Deformation in Circular Plates: A circular plate bent in synclastic curvature under symmetrical loading and restrained at its center acts as a circumferential cantilever. The radial fibers are elongated on the convex side of the neutral surface and are shortened or compressed on the concave side of that surface in proportion to their distances above or below it. The circumferential fibers also suffer corresponding elongations and contractions depending on their distance from the center of this area and the curvature of the radial fibers.

Referring to Fig. 28 the position of the plate bent downward at and near the center is shown in an exaggerated manner. The elongation of the top radial fiber of the plate as compared with the corresponding radius R of the neutral axis is noted as ΔR. The plate may be regarded as made up of horizontal thin sheets or laminas each of which consists of flat rings that are uniformly stretched in all directions. The work of stretching may be investigated in the following problem in which R is the radius of the circular sheet, ΔR the amount of the radial stretching of R, r is the radius of any ring of the sheet, Δr its radial stretching and $\Delta\theta$ the elementary angle between two consecutive radii.

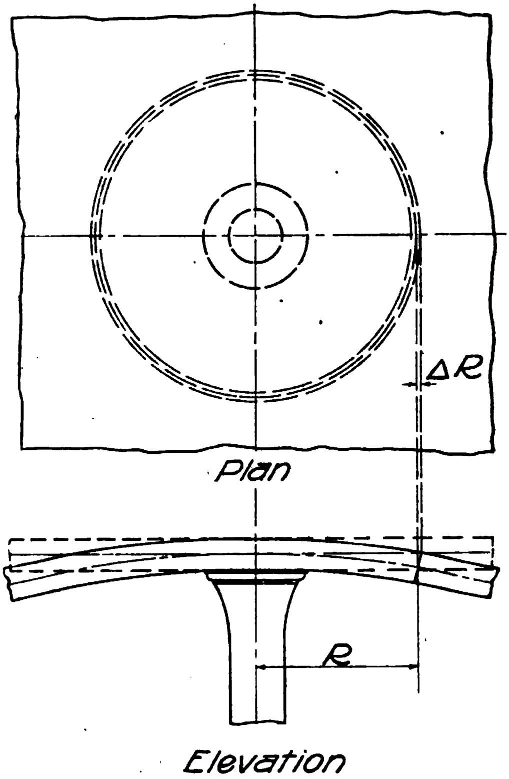

Fig. 28

Problem: In case of uniform stretching of a thin flat circular sheet taken as one unit thick, to compute the radial and the circumferential work against the elastic forces having a final intensity of f per square unit of cross section.

Resilience = (mean unit stress) × (cross section) × (elongation).

Mean unit stress during deformation $= \frac{1}{2}f$.

a. *Radial Resilience:* By definition of modulus E,

$$R : \Delta R :: E : f, \therefore f = \frac{E\Delta R}{R}$$

Mean cross section of elementary sector $= \dfrac{R\,d\,\theta}{2}$

Hence the radial resilience of an elementary sector

$$= \frac{E\,\Delta\,R}{2R} \times \frac{R\,d\,\theta}{2} \times \Delta\,R = \tfrac{1}{4}\,E\,(\Delta\,R)^2\,d\,\theta.$$

Total radial resilience of circular plate

$$= \frac{E\,(\Delta\,R)^2}{4} \int_0^{2\pi} d\,\theta = \frac{\pi}{2}\,E\,(\Delta\,R)^2$$

b. *Circumferential Resilience* of elementary ring of radius r.

By geometry $2\,\pi\,r : 2\,\pi\,\Delta\,r :: 2\,\pi\,R : 2\,\pi\,\Delta\,R :: E : f, \therefore f = \dfrac{E\,\Delta\,R}{R}$

Hence the circumferential resilience of elementary ring

$$= \frac{E\,\Delta\,R}{2R} \times d\,r \times 2\,\pi\,\Delta\,r, \text{ but } \Delta\,r = \frac{r\,\Delta\,R}{R}$$

∴ Total circumferential resilience of circular plate

$$= \frac{\pi\,E\,(\Delta\,R)^2}{R^2} \int_0^R r\,d\,r = \frac{\pi}{2}\,E\,(\Delta\,R)^2$$

∴ Total circumferential resilience = total radial resilience.

While the circumferential deformations are coincident with and their magnitudes are dependent on radial deformations it is the radial deformations or extensions and compressions along meridian lines which determine the vertical geometry of the slab. Since half of the work done by the load is stored in the plate circumferentially in a manner which does not enter into the determination of the deflection we have established a method of storage of energy in dish shaped curvature of the plate different from that in beam action. In a linear cantilever or in simple beam action no energy can be stored circumferentially because there is no elastic lateral deformation thru which to store energy laterally.

Let a uniformly distributed load W_1 when gradually applied to a rectangular plate of length $=2R$ and width $=\frac{1}{2}\pi$ in the form of a double cantilever or wide beam which is supported on a wall across the middle of its length perform an amount of work $=Q$ by producing a mean deflection H_1, then $Q=\frac{1}{2}W_1H_1$, and this amount of energy is stored in the plate in elastic deformations produced by the bending consisting of elongations and compressions all of which are lengthwise of the plate.

Again, let this same amount of work Q be performed upon a uniform circular plate of radius $=R$, which has the same length of span, the same area, the same thickness and the same amount of resistance per unit of cross section as the rectangular plate and is restrained at its center. Let W_2 be the load required to do this and let H_2 designate the mean deflection produced. Then $Q=\frac{1}{2}W_2H_2$. Now since one half of the energy of the tensions and compressions in the circular plate is stored circumferentially and one half radially and the radial deformations shown by the bending of the meridian cross sections of the plate are alone operative in producing the deflection H_2, we have on the average only about one half as much radial deformation of elongation or compression at a given distance from the center of the circular plate as occurs at that same distance from the middle of the rectangular plate. Considering the fact that the circular plate will on this account have a mean deflection H_2 which is approximately only one half of that of the rectangular plate, we have $H_2=\frac{1}{2}H_1$, approximately. But since in that case $Q=\frac{1}{2}W_1H_1=\frac{1}{2}W_2H_2$, we have $2W_1=W_2$. From this it appears that we can store approximately the same amount of energy Q in a circular cantilever plate of radius R supported at its center as in a double simple cantilever of width $\frac{1}{2}\pi R$ the length of each of the equal arms being R when the cross section of each plate is the same per unit of width by placing twice as much total uniform load upon the circular plate as upon the other and thereby producing half as great a mean deflection.

Now for purposes of more exact comparison let us no longer assume that the energy stored in the rectangular plate and in the circular plate are equal, but instead let the mean deflections be equal. Retaining the symbols $W_2=2W_1$, and $H_2=\frac{1}{2}H_1$, with significations with which they have been already employed let W_2' and H_2' be the load and deflection in case the deflection of the circular plate is equal to that of the rectangular, *i. e.* $H_2'=H_1=2H_2$. But to produce a deflection $2H_2$ will require twice the load required to produce the deflection H_2, i. e. $W_2'=2W_2'=4W_1$. And furthermore, since with equal deflections the longitudinal strains are practically equal the stresses are so also. Hence it requires four times as much load upon a circular plate to produce given stresses as upon a rectangular plate of equal area equal span $2R$ and like cross section per unit of width.

The foregoing demonstration omits all consideration of the lateral strains due to longitudinal stress, such as is required in obtaining the true stresses. Correcting this omission, the relative resistance and stiffness of the circular plate of homogeneous material as compared with the plate of equal area and unit cross section bending in beam curvature is the product of the foregoing relation by the reciprocal of $1—K^2$. If K is one half, the circular plate would be five and one third times as strong and five and one third times as stiff as the wide beam having the same quantity of material.

To apply this deduction to floors of Type IV, we may consider the column supported slab as divided into circles having a diameter equal to half the column spacing diagonally. These circles would then represent the areas of inverted dish shaped plates concentric with the columns and a suspended circular plate concentric with the center of the panel concave upward with saddle shaped areas between them. This analysis will give approximate values for the relative stiffness and stresses because so far as the stresses and strains in the concave suspended circle at the middle of a panel is concerned the four columns at the corners furnish a support which is nearly as symmetrical with respect to it as would be furnished by a circumferential support at the same distance, and a similar symmetry exists about each column supported convex cantilever circle. Such an arrangement, however, does not present a complete theory taking into consideration the action of shears and moments thruout the whole plate.

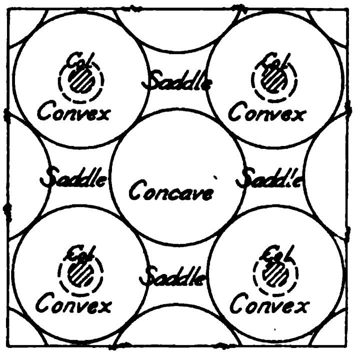

Fig. 29

The presence of saddle shaped areas shows that the resisting moment balancing the external moment of the applied forces comprises a combination of both twisting and bending moments which are to be separated from each other. Under uniform load in an interior panel we have by symmetry, eight lines of zero shear, two diagonal lines, four lines passing thru each of the four columns supporting the panel and two lines parallel with the column lines at right angles to each other intersecting at the center. The panel is thus divided by lines of zero shear into eight parts which may be treated independently in determining the applied bending and twisting moment of the load about the column supports as will now be shown.

23. Statical Demonstration of the magnitude of the bending moment parallel to each edge of a square panel of a column supported flat plate. In Fig. 26 let O be at the middle of a panel and D at the center of a column cap of which HJK is the quarter of the perimeter lying within the quarter panel $OADB$.

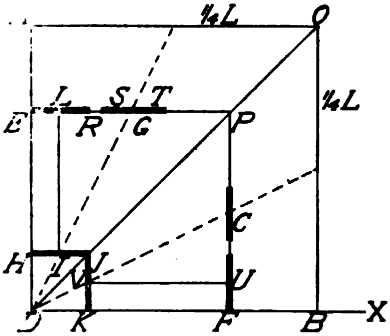

Fig. 30

In case of a total uniform load of W pounds per panel of a continuous uniform plate of many panels, since by reason of symmetry the diagonal OD as well as all four sides of $OADB$ are lines of zero vertical shear at all points, the entire load S resting upon the trapezoidal area $OAHJ$ is wholly supported by the vertical shears at the edge HJ of the cap, and the equal load S on the area $OBKJ$ by the shear at the edge KJ.

Similarly the vertical shear at any section PE parallel to OA is equal to the load upon the area $OAEP$ lying between PE and the median line OA.

Let G be the center of gravity of the area $OAHJ$, then DG bisects both PE and OA and all sections parallel thereto, and the resultant shear S due to the entire load $W/4$ upon $OAHJ$ acts at G, but is transferred from G and applied to HJ by help of the statical moments produced in the plate by this transfer.

In order to determine precisely what these moments are, assume lengths ST and ER equal and parallel to HJ, and for convenience let the resultant shear S at G be uniformly distributed along ST. Also assume that two equal and opposite shears equal to S act on ER. They will in no way disturb the equilibrium already existing because they themselves constitute a system in equilibrium and may consequently be arbitrarily introduced into the system previously existing for the purpose of demonstration, if so desired, without affecting results in any way.

Now designating upward shears applied to *OAHJ* as negative and downward as positive the shears and moments acting on *OAHJ* may be enumerated in pairs, as follows:

1st. A negative shear S at HJ and a positive shear S at ER, these together transfer the reaction S of the support from I to L and in doing so produce a bending moment $= S \times IL$ about OA or BD or some other axis perpendicular to IL.

2nd. A negative shear S at ER and a positive shear S at ST, these together transfer the supporting reaction from L to G and hold the resultant load S in equilibrium at G and in so doing produce, (since they are equal and opposite shears in the same plane) a twisting moment $= S \times LG$ about AD or OB or some other axis perpendicular to LG.

Similarly the load upon *OBKJ* produces a positive resultant shear S at C the center of gravity of this load at the mid point of PF, and this with the negative shear S at KJ produces a bending moment $= S \times VU$ about some axis as AD, perpendicular to VU and a twisting moment $= S \times UC$ about some axis as BD, perpendicular to UC.

But $LG = \frac{1}{2} VU$, and $UC = \frac{1}{2} IL$. Hence the twisting moment about either axis perpendicular to a side of the panel is one half as great as the bending moment about the same perpendicular, and the load S upon each of the trapezoidal areas *OAHJ*, *OBKJ* produces a bending moment about an axis parallel to one side of the panel and a twisting moment about an axis perpendicular to that side.

In case the column cap is reduced in size to a mere point at D each trapezoid becomes a triangle of area $= \frac{1}{8} L^2$, and $LG = UC = L/6$, while $IL = VU = \frac{1}{3} L$. Hence the bending moment M_x reduces in this case to $WL/24$ and the twisting moment to $WL/48$. With the ordinary size of column caps the moment each way reduces to $WL/16$ and sometimes with a large cap to as low as $WL/20$ as will later be developed. Similar results hold true in case the panel is not square.

Remarks: The total statical moment about a side for the entire load on a half panel between a median line and a parallel panel edge is twice the sum of both the bending moment and the twisting moment for a quarter panel. This sum amounts to $WL/8$, which statical moment must be held in equilibrium by the resistance of the combined bending and twisting moments that act in the material of the plate about the side in question.

It has been heretofore erroneously assumed without proof that the entire amount of both of the statical moments about x and y due to the loading must be all expended in producing bending moments alone about x and y respectively in flat plates just as in beams.

The nature of this error may perhaps become apparent by considering a square flat plate of dimensions $=10L$, on each side, let us say, supported by a series of parallel walls with flat tops at a distance apart of L along x. If this plate be uniformly loaded it will constitute a wide beam of ten spans along x subjected to bending moments M_x determined by the applied statical moments, while all moments M_y are zero.

If now the walls be replaced by separated supports at a distance apart $=L$, then the applied statical moments along x will be unchanged but statical moments along y of equal amount will also be introduced. Now since these applied statical moments in this case produce twisting moments also, such as did not exist with the wall supports, the bending moment M_x is less than with the wall supports, while M_y has at the same time been increased from zero to an equality with M_x.

From this it is evident that while the total bending moment about a single side is less with separated supports than with wall supports on two opposite sides of the panel, yet the sum total of the moments, M_x, and M_y in two directions, which are together effective in transferring the load to the supports, is greater than is the case with walls where $M_y=$ zero.

This matter has been so commonly misunderstood that the Joint Committee of the American Society et al. accepted Mr. Nichol's* erroneous analysis of the steel stresses in flat slabs in their report. From the foregoing demonstration it is apparent that in practice these bending moments are less than many have assumed them to be. They are in fact less than half of $WL/8$.

24. Equilibrium of an Elementary Vertical Prism Under Bending Moment and Twisting Moment. If in Fig. 30. on a previous page, AO be divided into equal parts and radial lines be drawn from the point of support D to each of these points of division as in Fig. 31, each pair of consecutive rays will bound a triangular area such as N_1DA, N_2DN_1, etc., having a common height and base and therefore of equal area though different in configuration.

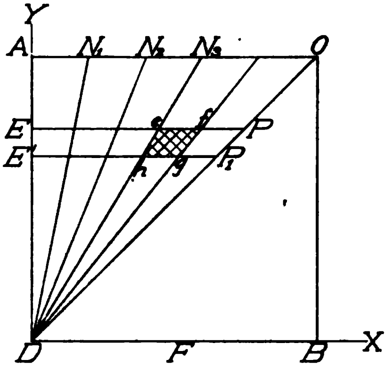

Fig. 31

Each of the radial sections thru D is a section of zero shear at every point and the load upon any trapezoidal area

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*Trans. Am. Soc. C. E. Vol. LXXVII, p. 1670 (1914).

such as that between *ef* and the corresponding segment of *AO* produces a vertical shear at *ef* equal to that load.

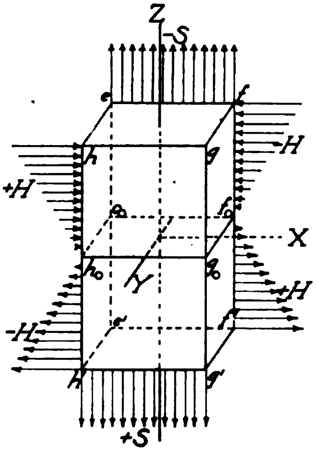

Fig. 32

Designate its amount per unit length of *ef* by $+s$, then the downward applied vertical force exerted on the element *efgh* at *ef* by the load between it and *AO* differs by no more than an infinitesimal amount from the upward supporting force $-s$ exerted on it at *gh* by the triangular portion of the plate between *gh* and *D*. These forces are vertical shears of intensity $\pm s$ which are uniformly distributed along *ef* and *gh*.

It is to be observed that these shears on *ef* and *gh* are not perpendicularly opposite to each other, but on account of the obliquity of the rhombus *efgh* one shearing force is shifted laterally with respect to the other, at the same time that it is shifted perpendicularly to the plane on which it acts. The perpendicular shift produces bending moment, and the lateral shift twisting moment as may be seen from Fig. 32, where the vertical shearing resistance which the element *efgh* exerts at its back face *eff'e'* on the trapezoid between it and *AO* is taken as equal and opposite to that at the front face *ghh'g'* on the triangle *Def*. In this Fig. 32 it does not appear so clearly as on Fig. 31 that these forces are not directly opposite each other, but that may nevertheless be readily understood.

Vertical shear is uniformly distributed along any section *PE* parallel to *AO* as may be seen from the fact that the shears on equal segments are produced by equal portions of the load. It should be further noted that since the twisting moment depends upon the lateral shift of this uniform shear it increases uniformly from zero at *E* to its greatest value at *P* at the diagonal *DO*.

Now the twisting moment produced by this lateral shift of the vertical shears between front and back faces about a horizontal axis *Y* is such as would tend to cause the laminae parallel to front and back faces to twist around on each other about *Y*. But the couple furnished by horizontal shears $\pm H$ on the back face, and horizontal shears $\mp H$ on the front face will cause the same kind of twisting about *Y*. As already explained these horizontal shears in vertical planes are the effective resisting shears in plates so far as twisting moments about horizontal axes are concerned.

It will be observed that the horizontal shears $\pm H$ on that part of the front and the back face that is above the neutral surface furnish a twisting moment about the vertical axis Z that increases in magnitude from the top surface to the neutral surface, etc., and further that it is held in equilibrium by the opposite twisting moment produced by $\mp H$ below the neutral surface. These moments tend to make the prism twist about the axis Z. So far it does not appear that there is any necessary connection between these horizontal shears on vertical planes and the vertical shears on the same planes.

When the intensities of the shears are equal on vertical planes at right angles to each other they have equal and opposite twisting moments about a vertical axis and neutralize each other. But this need not take place. As already seen on a previous page twisting moments about a vertical axis may also be produced otherwise than by these shears, viz. by the lateral shifting of the longitudinal bending stress T. The relative amounts of resistance to the resultant twisting moment about a vertical axis per horizontal unit of area are adjusted between unequal horizontal shears on vertical planes at right angles and the twisting shears on horizontal planes by the relative rigidities of these two kinds of resistance. And it is by means of this adjustment that the longitudinal stresses of the bending moments are differently distributed in different cases. For example in the crown sheet of a boiler which is pierced for stay-bolts at points of support the stresses across the sides of the panels are different from the stresses in a plate not so pierced.

It is to be noticed that the preceding discussion does not require the intensities of the shearing components of the stresses on vertical planes which are at right angles to each other to be equal, because of the intensity of the twisting moments that may exist on horizontal planes. In this particular the equilibrium of a rectangular elementary prism extending from top to bottom of a flat plane differs from that of a horizontal prism extending thru from side to side of a beam, which last is so symmetrically situated with respect to the shearing stresses that no twisting moments can occur in it as they can occur in an element of a flat plate. For this reason, vertical and horizontal shearing stresses have equal intensities in simple beams but horizontal shears on vertical planes at right angles need not be equal in flat plates.

25. Summary. The monolithic nature of reinforced concrete has been taken advantage of by the practical constructor, but it presents problems that from the theoretical standpoint are so different from those of structures made up of independent units of steel or timber that the theoretical treatment adapted directly from that of unit construction has not proven to be in accordance with practical experience. The fundamental errors which have brought this result about are as follows:

1. The attempt to treat the external moment of the load, which is merely an applied moment in the abstract, as a concrete quantity, and to fix its status by calling it without warrant an external bending moment, meaning thereby that it must necessarily be opposed by bending resistances alone regardless of the presence of internal resistances other than those of bending. These other resistances that take part in holding the external moment of the load in equilibrium are internal twisting resistances which latter combined with the internal bending resistances together furnish internal resisting moments that hold the external applied moment in equilibrium.

2. Treatment of shear as though it were not a directed quantity with a definite relation to the moment which is produced by it as a by-product or an effect of the shear.

3. Treatment of moments in the column supported panel as tho the load were carried by beam action alone regardless of the coaction of radial and circumferential resistance.

It is to be regretted that the foregoing errors vitiate the conclusions respecting flat plate floors reached by the Joint Committee on Concrete and Reinforced Concrete in their final report.

CHAPTER IV.

MECHANICS OF PERMANENT DEFORMATION AND STRAIN BEYOND THE LIMITS OF ELASTICITY

1. Where an elastic body is subjected to stress, and the strains it undergoes do not pass beyond the limits of the true elasticity of the materials, the work expended in straining the body is restored by its resilience as the stress is removed. But when the strains exceed these limits, work is expended upon the material that is not so restored. The energy so expended beyond the elastic resilience of the body is dissipated through the evolution of heat, and the body is not able to restore the energy imparted to it by work performed beyond true elastic limits.

This work done upon the material tends to change its properties. When a piece of mild steel is gradually loaded in simple tension to between five-eighths and two thirds of its ultimate strength, a point is reached where a large plastic deformation occurs with very little increase in the applied load. This is called the **yield point**, or commercial elastic limit, and is determined by the drop of the beam in testing. When the load is carried to a point somewhat beyond this primitive yield point and the load is then removed, on subsequent testing of the same piece it will be found to have developed a new yield point at a somewhat higher load than that previously found, while the limit of proportionality between stress and strain has at the same time been reduced below the value previously found. If, after being considerably over strained the specimen be allowed to rest for a certain interval of time before being reloaded, it then exhibits greater hardness than does a similar specimen similarly treated and reloaded immediately. Its yield point is further raised by such a period of rest and its elongation at fracture is also thereby reduced.

The apparent limit of proportionality also rises during this period of rest from that position to which it was lowered by the over-strain. When a piece of steel is strained beyond the limit of true proportionality and is nevertheless within the yield point value, a temporary strain is observed after the removal of the load. The disappearance of the temporary strain which occurs by a slow creeping back of the material is called elastic after-working and the lag or slow increase of deformation under stress is called hysteresis.

When a bar of steel is loaded a number of times alternately in tension and compression to a point a little beyond the yield point value, the yield point values for tension and compression are thereby both reduced to about .6 their original value, called by Bauschinger the Natural Elastic limits of the material.

2. **Apparent Elastic Fatigue.** Experiments on the torsional vibration of wires detailed by Kelvin in his article on Elasticity in the Encyclopedia Britannica, indicate elastic fatigue for a considerable range of stresses through which the experiments were conducted, this range being within the limit of elasticity of shape. The experiments were as follows:

About April 26th, two equal and similar lead weights or vibrators were hung up by equal and similar copper wires, the upper and lower ends of the two wires being similarly fixed by soldering. Wire No. 2 was kept vibrating as continuously as possible from day to day, and the other kept at rest except when vibrated once a day in an experiment to test the comparative vicosity of the wires, but no exact comparison was made until May 15. Then No. 1 in 97 vibrations subsided from an initial range of 20 to a range of 10, while No. 2 gave the same subsidence in 98 vibrations. During the greater part of May 16 and 17, No. 2, was kept vibrating and No. 1 quiet and experiments made, with the result that No. 1 subsided from an initial range of 20 to 10 in 98 to 99 vibrations of 2.4 seconds each, while No. 2 gave the same subsidence in from 57 to 60 vibrations of 2.4 seconds each.

It was supposed from these experiments that true elastic fatigue occurred for all stresses within the limit of elasticity of shape until a similar and more extensive series of experiments conducted by H. Thomlinson (Phil. Trans. Royal Soc. 1886) showed that no such fatigue of elasticity is felt in steel, iron or copper provided the stress does not exceed a certain well defined limit.

3. **Rupture by Fatigue.** The behavior of iron and steel under fluctuating or repeated stresses was the subject of long research by *Wöhler and later by Bauschinger. The results of these experiments show:

1. That resistance to fracture under stresses fluctuating within certain limits depends upon the difference between the maximum and minimum applied stresses rather than upon the maximum stress alone.

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*Engineering. London. Vol. XI. 1871.

2. And that reversed stresses, tensile and compressive, much below the static breaking stress, are sufficient to cause fracture if repeated a great number of times.

Experiments upon axle iron made by the Phoenix Company, having an ultimate strength in tension of 23 tons with an elongation of about 20 percent, gave limiting maximum stresses under limited ranges as follows:

Kind of Repeated Load	Stress in tons per square inch	
	Limiting Maximum	Limiting Range
Completely reversed	7.6	15.2
From Maximum to Zero	15.28	15.28
From Maximum to Half Load	21.01	about 10

Now the ultimate strength of the material was 23 tons and its primitive yield point was in the neighborhood of 12 to 13 tons.

The limiting maximum stress of 21 tons for repetitions of half load in the above Table is therefore 8 tons above the primitive yield point, and since whenever the primitive yield point is exceeded a large deformation results, the manner in which these repeated stresses were applied is of interest.

To avoid the large deformation which would have been produced when the primitive yield point is exceeded, the stresses first applied in these experiments, were less than the primitive yield point and this point was thus gradually raised until its ultimate position was above the maximum stress considered in the range noted as the limiting range.

In a practical structure the conditions of loading are not such as to raise the yield point value of the material and so increase its resistance, as was done in these experiments.

Consequently the limiting range of intensity of stress which the practical structure can withstand cannot be deduced from experimental results where such artificial assistance is extended to the resistance of the material.

The number of repetitions of stress upon the specimen, still unbroken, from which these limiting ranges were determined, ran into the millions. The specimens unbroken at an alternating stress of 7.6 tons were subjected to 132,000,000 repetitions without breaking.

This limiting range which allows nothing for dynamic effect is nearly .6 of the primitive yield point value of the material, and if the dynamic effect of the rapidity of repetition of stress amounts to 15% the limiting range would be in the neighborhood of .7 of the primitive yield point.

4. **Dynamic Theory of Fatigue.** It is well known that the effect of a suddenly applied load in producing deflection is greater than that due to the same load at rest and that were the load imposed with complete suddenness without shock its effect would be to produce a momentary deflection and strain twice that of the same load at rest. Mr. Fidler in his treatise on bridge construction advances the view that it is permissible to doubt whether the apparatus used by Wöhler did in fact limit the internal stresses to the calculated amount without exceeding it, and whether the bars were not subjected to action of dynamic stresses every time the load was applied which took place about four times a minute. That this was the case to some extent can hardly be doubted, but what the limit of that effect was is the question to be decided if Wöhler's results may be interpreted on this basis.

A study of the Wöhler machines would lead the engineer to select the one for repeated bending as most likely to cause the maximum dynamic stress and perhaps his estimate of this effect in view of the slow speed of the machine, four revolutions per minute, would be 20 to 25 percent added load on the cantilever end of the specimen. Mr. Fidler's estimate for this would be six to ten times that suggested by considering it possible that the elastic vibrations induced might bring the internal stress to the maximum effect of a suddenly applied load which in this case would be three times the load.since the range of stress is twice the load. The following is his table for the maximum breaking load *a*, by the dynmaic theory put forward by him:

1. Steady load, no variation.................. $a = t$
2. Load varying from 0 to u.................. $a = u = t/2$
3. Load varying from $-v$ to $+v$.............. $a = v = t/3$

If a bar be subjected to a suddenly applied load, the external work of the load is its weight times the deformation of the bar, which must be balanced by the internal work, which is the product of the mean stress times the elastic deformation: Hence, the deflection, or deformation, for the suddenly applied load, will be twice that caused by the same load at rest, since the mean internal stress is only half the maximum. If, however, this dynamic stress exceeds the yield point, part of the internal work will be performed in plastic deformation, resulting in raising the yield point, just as with a steadily applied load. Hence, in ductile material, a suddenly applied load greater than half the ultimate strength would have to be applied a number of times to produce rupture, and on this basis Mr. Fidler undertakes to account for the endurance of specimens in the Wöhler experiments where the dynamic stresses would (if figured on his proposed basis) be considerably greater than the nominal strength of the material.

Now the yield point in mild steel is approximately 60% of the ultimate strength, and if we limit the range, plus and minus, of the alternate stresses, in the Wöhler machine to something less than one-third of the ultimate strength, the endurance of the piece seems practically unlimited, and the yield point has not been raised, as no plastic deformation occurs, although the dynamic stresses by Mr. Fidler's method of computation have exceeded the yield point by between 40 and 50 percent. Since the conditions of application of the load have been the same for the lower and higher intensities of stress, the same line of reasoning advanced to support the probable accuracy of the dynamic theory may be used with far better advantage to prove its improbability.

It has been argued that, in a short test piece under direct stress, the amplitude of the elastic vibrations under these dynamic stresses would be so small as to escape observation. This argument would indeed have some weight were the stresses direct and the test piece short, but as similar fatigue occurs under bending, where the length of the specimen would insure so considerable an amplitude of vibration as to render their detection certain if these stresses were of anything like the magnitude supposed, the argument would seem to be invalid.

While the dynamic effect of a load imposed instantaneously and without shock in producing deflection and stress may be shown mathematically to be momentarily twice that of the same load at rest, it is not possible to demonstrate such a relation at all if the time of application is extended through an interval of fifteen seconds. In fact this difficulty is so unavoidable as to render the theory plainly untenable.

Further experiments have shown that the number of repetitions required to produce rupture remains substantially constant whether the speed of application of the fluctuating stresses is decreased to half that in the Wöhler experiments or doubled. The dynamic theory therefore fails entirely to account for the phenomena in question.

5. Micro Flaw Theory of Fatigue. Johnson, in his "Materials of Construction" attempts to account for fatigue by micro-flaws in the material. If they are the usual cause of ultimate fracture in fatigue tests, we should look for their influence in the elastic fatigue of Thomson's experiments, but since in these the elastic fatigue observed depended on the molecular condition immediately preceding the observation whether of quiescence or recurring changes in shape, it cannot be accounted for by micro-flaws which are necessarily permanent and not temporary.

6. Work of Rupture in Steel. When a bar of steel is ruptured by a single application of load gradually increasing from zero to its ultimate resistance, but a small portion of the deformation is elastic, viz., that part which occurs before the yield point is exceeded. Permanent stretching then commences and continues until the bar may have elongated as much as 20 percent of its length. The mean stress causing the plastic deformation would be a mean between the yield point and its approximate ultimate value which for mild steel would be the mean between 38,000 to 40,000 and 60,000 to 62,000 or about 50,000 pounds per square inch. Consequently, the work absorbed by the material per cubic inch before reaching the approximate point of rupture, disregarding the elastic work which is relatively negligible, would be 50,000 pounds × .2, or 10,000 inch pounds of work or 833 ft. pounds $= 1.07$ of the heat required to raise a pound of water one degree F.

But this amount of heat is sufficient to raise the temperature of the cubic inch of steel something like 33 degrees F., an amount which is readily appreciated by the sense of touch and is always noticeable to the close observer even though part of it be dissipated by conduction and radiation.

When rupture occurs accompanied by nearly negligible ultimate deformation under repeated stresses of intensity lower than the ultimate carrying strength of the material, there appears to be no good reason to suppose that the total of the energy transformed into heat in small amounts under successive repetitions of the load in an endurance test producing rupture is any less than where this deformation is as large as it is in the case of a single application of a load which develops the maximum carrying strength of the specimen.

In fact, the general validity of the law of conservation of energy would seem to indicate conclusively that the same amount of energy must be expended to produce rupture of the same area of a given kind of material, whether it is expended in small quantities at different times or at one single application.

Since this transformation of energy in fatigue tests takes place in small increments and with inconsiderable deformation, the only way of determining experimentally the true elastic limit appears to be by following experimentally the energy expended in work on the material which makes its appearance in the form of heat.

The observations of Bauschinger that specimens of steel so far from being injured by repetitions of stress from tension to compression appear instead to be improved in elasticity, ductility and tenacity when the stresses are within the natural elastic limits, indicates not only that such limits of elasticity exist but also that they are lower than has been heretofore supposed from mere experimental determination of the apparent limit of proportionality and yield point of the material.

If we regard fatigue as a molecular phenomenon it is evidently desirable to have some apparatus for the measurement of such minute changes which is far more sensitive than the extensometer. Fortunately, suitable instruments for the purpose are to be found in any well equipped physical laboratory, that are 20,000 times as sensitive as the extensometer.

The fact that when the elastic limits are exceeded, work is done upon the body in permanently affecting its internal condition and that energy is then transformed into heat and dissipated by radiation leads at once to the consideration of the thermo dynamic relations of stress, strain and temperature.

7. The Mathematical Theory of the Thermo-Elastic Properties of Matter has been developed by Kelvin, (Quarterly Mathematical Journal, April 1855) and the following formula which is applicable to a metal bar is there demonstrated:

$$H=\frac{t\,p\,e}{J\,S\,W};$$

where H = thermal change in degrees Centigrade;

t = temperature, Centigrade, from absolute zero;

J = the mechanical equivalent of the thermal unit, = 1400 ft. lbs. per degree Centigrade;

p = pressure applied, in pounds, positive if compression and negative if tension;

e = the longitudinal expansion per degree Centigrade;

S = the specific heat;

and W = the weight, in pounds per foot of length of bar, This change in temperature, as would be inferred from the mechanical theory of heat, takes place instantaneously, with the application of the pressure, or load, p, and is subject to the usual laws of radiation. Since the temperature change is very small, Newton's Law of Cooling may be taken to apply exactly, namely; The amount of heat gained or lost by a body in a given time is proportional to the difference in temperature between the body and the surrounding medium. Hence if two loads, of different magnitudes, are gradually applied in equal periods of time, the temperature changes in the bar for the respective loads will be strictly in proportion to the loads, notwithstanding the radiation going on while the loads are being applied.

The formula

$H=\frac{t\,p\,e}{J\,S\,W}$ is rigidly exact for any solid only for an infinitesimal increment in p; but with steel, any variation from it within elastic limits would be extremely small, and may be taken as exactly proportional to the pressure or load applied, and, as we are interested only in relative measurements, no error is involved.

As H depends on t, the temperature of the bar at the time the load is applied, this temperature should be noted, if readings taken at different temperatures are to be compared. Thus, if the value of absolute zero be taken at —276.7° Cent., and one set of measurements is taken at 23.3° Cent., and another at 24.3° Cent., the latter, for the same loads, should be 1/3 of 1% greater, if no error has been made in the determination. As t would probably not vary greatly, this percentage per degree Centigrade is sufficiently accurate for most purposes.

The magnitude of the temperature change, H, may be best illustrated by a few figures. Take the temperature of a bar at 19.3° Cent., p at 1 lb., and the area of the bar at 1 sq. in., and we have, for mild steel,

$H=\frac{(276.7\times 19.3)\times 10\times .00001142}{1400\times 0.116\times 3.34}=0.00000623$ Cent., the temperature change for a change in stress of 1 lb. per square inch.

Such minute temperature changes as those resulting from the application of a load of even one quarter ounce per square inch may be measured by thermo electric means and it is therefore possible to determine the range of true elasticity of such a homogeneous material as steel by this method with a high degree of precision. Such measurements and the apparatus employed have been described by Turner, (Trans. Am. Soc. C. E., August 1902). The apparatus used was an antimony-bismuth thermopile, insulated from and applied to the polished surface of the specimen, in circuit with a D'Arsonval type of galvanometer with the usual scale and reading telescope for determination of the relative intensity of the current produced by different loads.

Suitable insulation of the thermo-pile from the animal heat of the observer, from radiant heat of electric light, and disturbing air currents is essential as well as the insulation of the pile to prevent any short circuit from contact with the polished surface of the bar. Material for the latter insulation was the subject of much study, rubber cement proving the most available material since it gave an exceedingly thin coat which did not interfere with the transmission of heat and effectually prevented short circuit.

It was found necessary to vibrate the scale beam of the testing machine in order that the weighing of the load should correspond with the degree of accuracy of the electrical measurements. From these experiments Turner shows that the limit of thermal proportionality in mild and medium steel is approximately five eighths of the yield point value of the metal. The bar was found to grow gradually cooler under increased loading until this limit was reached when the heat generated by internal molecular friction off-set the increment of reduced temperature and the thermal curve deviated from a straight line more and more until the yield point value was reached when the sudden evolution of heat would make the reading fall beyond the scale.

So in compression tests. The readings were found to be proportional up to approximately a similar limit when the increase in temperature of the bar would no longer follow the right line changes but would increase more and more rapidly as the yield point was approached but at that point the reading would be too great to fall within the limits of the scale.

8. Irrational Nature of Maximum and Minimum Formulas. If all the different phenomena presented by the rupture of material under repeated stress can be accounted for on the basis of the theory that such rupture is only brought about by stresses beyond the true limits of elasticity then the formulas of Launhardt, Weyrauch, Johnson and Merriman, et al, for rupture under repeated stresses are all mere algebraic misconceptions, wasteful in their demands for excessive section under ranges of stress which should improve rather than injure the material when applied without shock, and dangerous where the range of stress is small, leading to the use of stresses so high that no margin of safety remains within the limits of true elasticity.

Before the view advanced above becomes acceptable its implications and applications must accord with all known phenomena. The energy expended in rupture is greater for soft steel than for hard steel because of the greater plastic deformation in the soft steel. Consequently the theory indicates a more rapid destruction of hard steel than soft steel under all stresses exceeding the true elastic limits of the material, and this inference is substantiated by the experiments of Wöhler, Bauschinger, and others.

Again, because the true limits of elasticity are higher with hard steel than with soft steel the limiting range of indefinite endurance under fluctuating stresses from tension to compression should be higher with hard steel than with soft steel just as the experiments show the case to be.

Again, because no work could be done on the material when the range of stress does not exceed the true elastic limits, indefinite or unlimited endurance should be found under alternating stresses from tension or compression which are well within these limits. This accords with Bauschinger's conclusion that so far from being injured the material is improved in elasticity, ductility and tenacity under such range of stresses unless there are apparent flaws or defects in the specimen.

The range of thermal proportionality like the limit of proportionality of shape is decreased by any loading which exceeds the yield point value and it may be questioned whether it rises as readily during the period of rest as does the apparent limit of proportionality of shape. The fact that this thermal limit is reduced by loading which goes beyond the primitive yield point of the metal accounts for the more rapid destruction of the piece under stresses that alternate from tension to compression and exceed the yield points.

In order to account for the more rapid rupture of the material where the stresses range from tension to compression than where the range of stress is of one kind only, it is necessary in addition to the foregoing developments to consider the kind of effect produced by over strain upon the material and whether this effect is the same when brought about by tension as by compression.

Thomson found that the thermo-electric property of iron bars hardened by longitudinal compression is the reverse of that discovered by Magnus in bars hardened by drawing, and that iron under lateral compression manifests the same thermo-electric property that occurs in an iron wire under a longitudinal stretching force. As temporary strain changes the thermo-electric quality of the material so also does the permanent strain or set, but the residual effect of the permanent strain is the reverse of that which subsists as long as the restraining force is kept applied.

When the repetition of severe stresses are of one kind only it may be conceived that each repetition produces a change with the ultimate result that the life of the piece would be long under Wöhler treatment. But on the other hand, where the range of stress is from tension to compression, since the residual effects of the two kinds of severe stress are of opposite kinds, a correspondingly shorter life, due to the lack of similarity in the change in structure by the two kinds of severe stress would be expected, as was in fact found in the tests of Wöhler, and others.

It has been observed that the temporary effect of elastic strain is to alter the thermo-electric property of the metal and this has been attributed to an unequal distribution of inter-molecular forces opposing the external forces. Arguing, then, that similar effects are produced by similar causes, these residual effects may perhaps be attributed to a condition of permanent internal strain along certain lines or axes, the natural result of which would be to enable it more readily to resist stress of the kind which produced this condition, while at the same time it is rendered weaker in other directions, or in other words, less elastic or resilient and more likely to be fractured by shock or sudden loading.

The temporary alteration in thermo-electric properties of a given bar by elastic stress makes it possible with sufficiently delicate apparatus to measure the amount of dead stress in a piece of steel by comparing it with a sample of the same material unstrained, by forming a junction between the strained and unstrained metal which may be heated or increased in temperature a given amount, thereby producing a thermo-electric current when connected in circuit, and the intensity of this current may be measured with the usual apparatus. A galvanometer with very low resistance, however, is required in order to make such measurements possible.

Any sound theory of fatigue of material must be in full accord with all known phenomena. The dynamic theory, the theory of micro-flaws and the fatigue formula all fail alike to measure up to this standard. Consequently the rational treatment of range of stress applied to buildings or bridges should not be based upon such reasoning but upon a consistent theory of structural vibration which will be postponed for the present.

9. Equilibrium of Residual Strains. Residual strains are of two kinds temporary, which disappear with time, and permanent which do not disappear except when the state of strain is altered by external force or some other change of external conditions. True or elastic strain merges into temporary strain as the limit of proportionality of shape is approached and exceeded and temporary strain merges into permanent strain as the stress producing the temporary strain is increased more and more. Temporary strain is a state of imperfect equilibrium and because of the imperfect balance of the internal stresses therein, progressive changes of shape occur, and elastic after-working proceeds until equilbrium is restored.

Permanent strain is a state of internal stress in equilibrium from which no change in shape results unless this state of equilibrium is disturbed by the action of forces which supply energy from without. This energy which disturbs or changes the internal state of equilibrium may be introduced as follows:

1. By a change of temperature.
2. By an applied force.
3. By cutting away a portion of the symmetrically strained body.

Such removal or cutting away of symmetrical portions of an overstrained elastic material permits the permanent strains not in equilibrium to reassert themselves and change the form of the body in a manner comparable to applied external force.

Thus in a piece of cold rolled shafting the surface has been symmetrically strained by the work expended upon it. The removal of the material from a key way cut along the length of the shaft destroys the symmetry and equilibrium of the state of permanent internal strain and permits the permanent strains on the side opposite the key way to reassert themselves in a manner comparable to external forces, resulting in a longitudinal curvature or bending of the shaft until equilibrium is re-established.

In like manner, where a reinforcing bar has been bent cold, a state of permanent strain has been induced in the bar by permanent tension strains on one side and compression strains on the opposite side.

When a section of the fibers permanently strained in tension by such bending of the bar are cut away, the permanent compressive strains reassert themselves on the opposite side of the bar and tend to increase the curvature of the bar. Likewise, where a section is cut away on the opposite side of the bar, removing the reaction of the fibers which have received the greatest permanent strain under compression, the permanent tensile strains reassert themselves and likewise increase the curvature of the bar.

The fact that residual strains are opposite in character to elastic strains was demonstrated by simple experiment, as follows: Several ⅜″ square, mild steel bars were bent cold to a 6″ diameter as shown in Fig. 33 so that the spread of the ends was approximately 5¾″. The convex surface of one bar was scored with a saw with kerfs about ⅜″ spacing on the outside and to the depth of 3/16″. The distance between the ends of the curved bar decreased 1/16 of an inch by this process.

An equal and similar specimen bent in the same way was then treated by sawing kerfs on the inside of the bend. The curvature then changed a full 3/64 inch in the same direction.

Another and similar specimen was treated by sawing kerfs on the outside and inside ⅛″ deep only and of similar spacing. The change in curvature produced by these cuts was in the opposite direction from that of the single set on one side and amounted to 7/64″ decrease in curvature or spread of the ends of the horseshoe bar instead of increase in curvature. Now the residual strains in a bar bent cold in this form would present the greatest intensity at the outermost fibers and in the center would be an area where true elastic strain would still subsist. The elastic strain being the reverse in kind and character to that of the residual strain its effect is to straighten the bar out and reduce its curvature, whereas the residual strain tends to increase its curvature when thrown out of equilibrium.

This kind of operation is demonstrated by the reduction of curvature and the bending outward of the ends of the horseshoe 7/64″ by the elastic strain at the middle of the bar.

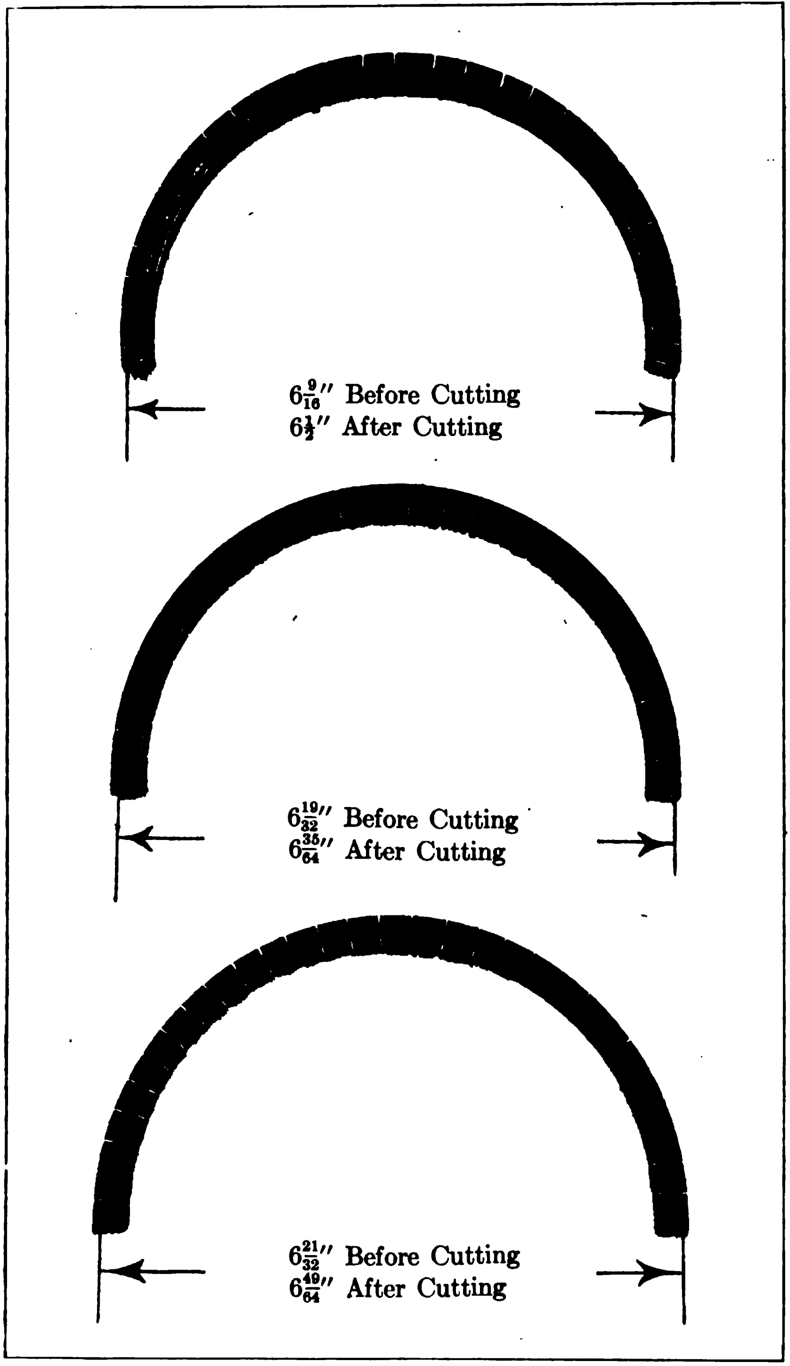
$6\frac{9}{16}''$ Before Cutting
$6\frac{1}{2}''$ After Cutting
$6\frac{19}{32}''$ Before Cutting
$6\frac{35}{64}''$ After Cutting
$6\frac{21}{32}''$ Before Cutting
$6\frac{49}{64}''$ After Cutting

Fig. 33. Experiments With Cold Bent Bars.

Examining the change in form of the bar brought about by cold bending the convex outside face was reduced in width .017″; the inside was increased .010″. Consequently it is apparent that the neutral surface of the residual strain was thrown toward the compression side and therefore the increase in curvature of the bar was larger in amount when the saw kerfs were cut on the outside of the curve than when they were cut on the inside of the curve.

The active nature of residual strain thus manifested affords a clearer insight into the phenomena of rupture under Wöhler treatment and of final rupture under loads of a small fraction only of those repeatedly borne thru the accumulation of residual strains of opposing nature. The relief of residual tensile or compressive strains by the saw kerfs cause curvature in the same direction because each tends to produce rotation in the same direction about the neutral axis.

10. Equilibrium of Axial Stress Superposed Upon a Balanced State of Residual Transverse Strain. Suppose a bar by cold bending is brought into a state of balanced residual strain similar to that of the specimens discussed. Because the Poisson effect in plastic deformation is to expand the compressed area and contract the area comprising the fibers that are stretched the neutral plane shifts from the center of gravity of the section, and the extreme fibers in tension are stretched more than those in compression are shortened, hence when an axial stress is brought upon the bar a state of unbalanced equilibrium is at once produced with the reasserted action of the residual strains accompanied by a deformation commonly mistaken for elastic strain in extensometer tests. Any local bending in a bar induces transverse strains that may cause a measurement which is made on one side to differ materially from the average of the measurements from opposite sides, which last represents the axial stress.

It therefore appears that when a concrete floor is initially loaded, and extensometer measurements are made upon bars which have been bent cold and are in a state of permanent strain resulting therefrom, the measurements show the effect of both the elastic strain and the reaction of permanent strain combined therewith.

The applied force of the loads by changing the form of the bar disturbs the primitive state of equilibrium of permanent strains within the bar, with a corresponding modification of strain-gage readings which are commonly mistaken for true elastic readings.

These two effects must therefore be separated from each other before the measurements can be regarded as possessing definite value from the standpoint of scientific precision.

11. Modulus of Rupture and Theory of Elastic Failure. Where a material, such as concrete or cast iron, whose tensile resistance is inferior to its compressive resistance, is subjected to gradual increased bending stress, sufficient to cause rupture, the phenomena of over-strain are accompanied by a shifting of the neutral surface. This change in position of the neutral plane from the center of gavity of the section (its fixed position for elastic deportment) results in more severe stresses being imposed upon the fibers in compression and relatively less stress upon the fibers opposing tensile deformation, *i. e.*, since for equilibrium the sum of the horizontal compressions must equal the sum of the horizontal tensions, shifting the neutral plane increases the maximum fiber stress on the side toward which it is shifted.

The intensity of the computed extreme fiber stress which would support the load at rupture when operating according to elastic theory, is called the modulus of rupture. The magnitude of this computed fiber stress is found to be approximately a geometrical mean* between the ultimate compressive and tensile resistance of the material considered, under bending stress produced by uniform loading.†

But the ultimate ‡resistance of concrete under restrained shear stress (punching shear) is also a geometrical mean between the ultimate tensile and compressive resistance of the material. Apparently this identity of relation between the modulus of rupture and the ultimate resistance to rupture by shear suggests an explanation of the varied phenomena of rupture, upon the theory that elastic failure is brought about by a certain maximum shearing stress, this being proportional to the greatest difference between the principal stresses.

The wide divergence in the location of initial cracks in reinforced concrete beams, which occur toward the end of the deep concrete beam and toward the center of the shallow concrete beam, with the same percentage of reinforcement, points to the foregoing explanation rather than to that of the magnitude of the maximum principal strain. Experiments bearing on this question have been published by J. J. Guest, and others, (Phil. Mag. July 1900; Proceedings Physical Soc. May 1900). Morley, Strength of Materials, page 57 says:

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*A geometrical mean between two quantities is the square root of their product.

†The moduli of rupture, Tests of Concrete Beams, U. S. Geological Survey, Bulletin 334, are less than the geometrical mean of the tensile and compressive stresses because the loads were applied at the third points producing sharper bending and reducing this normal value.

‡See Concrete Steel Construction by Morsch, Eng. News, Ed. p. 34, 35.

> "The theory has been advanced that all fractures under tensile stress are ultimately fractures by shearing under the tangential component stress, but this conclusion seems difficult to accept in the case of hard steel which does not show any trace of oblique fracture."

In soft steel we have the necking down or drawing in accompanied by reduction of area as compared to fracture without such reduction in harder steel. The tenacity of the material figured on the cross sectional area at the moment of rupture, appears however to remain nearly constant. Hence, if the material be regarded as made up of molecular groups or crystals cemented together in a matrix the effect of the increase in carbon appears to be that of hardening the matrix which prevents the sliding shear distortion. This conception of the matter is not inconsistent with the idea of shear failure in the hard steel, since such failure might be conceived as taking place along the faces of the minute crystals which because of the added rigidity of the matrix, are prevented from sliding, but not from pure shear deformation.

In investigation of the behavior of reinforced concrete when undergoing tensile deformation on the under side of the beam the experiments of Considère have been put forward as indicating that the concrete when reinforced elongates more before final rupture than when not reinforced and that the resistance of the concrete is nearly constant at its maximum value for some time previous to rupture. The known phenomena of static fatigue, however, indicate that such resistance other than that of the steel is in fact temporary and tends to disappear with continued application of the load.

In experiments of this sort it is of course difficult to determine when the concrete commences to crack because there is some plastic yielding and moreover the steel forces it to elongate with greater uniformity than when unreinforced, so that not only do the cracks open up more slowly because of this uniformity of distribution of elongation but their initial inception is also more difficult to detect.

The experiments by the Bureau of Standards and other published tests of reinforced concrete beams show that cracks, appeared in the concrete of beams having the particular ratio of depth to span adopted in these tests.at an elongation corresponding to five to six thousand pounds per square inch stress in the steel.

The method was adopted of detecting these cracks by water marks as first used at Wisconsin University. Beams cured in water were only partially dried before testing. When tested, preceding the appearance of the crack a dark wet line appeared across the beam. The condition of the partially dried out beam would not be strictly comparable to a thoroughly cured specimen. Nevertheless there does not appear to be as great discrepancy in the results from this cause as might be expected from the effect of partial drying and the shrinkage therefrom.

Now beams are usually continuous in the practical structure and are relatively stiffer and in general of greater depth in proportion to the length of the span than these experimental beams.

Accordingly in drawing any conclusions from these experimental beams and applying the conclusions to beams in practical building the difference in the stiffness due to continuity and greater relative depth of the section must be considered because cracking is brought about by a certain limiting shearing strain instead of by a limiting principal strain represented by the elongation itself. On this basis the fact that cracking occurs in the test specimen at an elongation corresponding to five or six thousand pounds tensile stress on the steel while in the continuous beam in the ordinary building, having a depth of one tenth of the span length and a positive curvature extending part of the length only, because of the great difference in the radius of curvature, the shear deformation is relatively much smaller in the practical beam than in the test specimen.

Consequently at the point where cracking commences from flexure the elongation is several times greater than in the test beam.

In discussing modulus of rupture, we have noted that under uniform load its value is greater than under load concentrated at the third points. The sharpness of curvature under concentrated load brings about the marked effect noted, again demonstrating the fact that rupture depends on a certain limiting shearing stress rather than upon a limiting principal strain.

In order to bring out clearly the tendency of different lengths of radii of curvature to produce limiting shearing strains in cases where the extreme elongations are identical, consider a beam of uniform section freely supported at its ends and loaded at the third points with equal loads. In the center third the vertical shear is zero. There is uniform bending and the curvature is a circular arc, the moment of inertia being constant.

Consider two such beams, one half as deep as the other. Suppose the elongation of the extreme fiber in tension to be the same in each. Then the rate of increase of this elongation brought about by horizontal shearing stress is twice as rapid in the shallow beam as it is in the beam of double depth and the maximum horizontal shearing strain between any two consecutive horizontal layers is twice as great in the shallow beam as it is in the deeper beam. Therefore if cracking of the concrete occurs at a certain limiting shearing strain instead of at a limiting principal strain we should have the cracking of the concrete in the shallower beam under an elongation much less than would occur in the deeper and stiffer beam, as experience proves the case to be. This matter will be discussed more at length in the mathematical theory to follow, see Fig. 86, Chapter VIII.

12. Static Fatigue. If we suppose a definite stress applied to a piece of steel or a certain weight placed on the middle of a steel beam simply supported, and applied and removed a great many times with the result that after each application and removal of the load the specimen comes to rest in exactly the same configuration as it had previous to the application of the load and if the specimen be left to itself with stress removed and it be found to remain in the same configuration for days or years after removal of the stress a part of the definition of perfect elasticity is fulfilled.

Or again, if the stress be applied and kept applied, and the specimen remains permanently in constant configuration another condition of the definition of perfect elasticity is fulfilled.

In many specimens of mild steel, instruments of great precision reveal slight permanent extensions resulting from very low stresses, particularly where these specimens have never before been subjected to such tensile stresses. From the disclosure of such permanent set under low stresses the erroneous conclusions might be reached that the limits of nearly perfect elasticity of the material were no greater than that at which these small sets occurred, but our present knowledge of the operation of residual strain accounts for this phenomenon on the ground that the set represents the relief of primitive residual strain induced by cold bending or other work done upon the material previously. After the relief of such residual strain a sound specimen of homogeneous steel or of good wrought iron exhibits nearly perfect elasticity of shape and configuration up to approximately eight-tenths of the yield point value, after which noticeable hysteresis or lag occurs in the development of the final deformation under a given carrying load. This slowly increased deformation beyond the limit of proportionality grows less and less as the time is increased by hours or days and the time stress curve becomes asymptotic in a relatively short period of time.

Experiments beyond the yield point show similar phenomena, and it seems highly probable that a perfectly quiescent load exceeding the primitive yield point might be applied for an almost indefinite period without producing ultimate rupture in sound metal. The product of the faggot mill which is mere muck bar and steel scrap rolled together will frequently exhibit increasing deformation under constant load of intensity of half the yield point value of the metal for a considerable period of time. Such metal is most undesirable as a reinforcing material and the same statement can be made regarding the poorer qualities of rerolled stock or any metal which has been subjected to improper heat treatment.

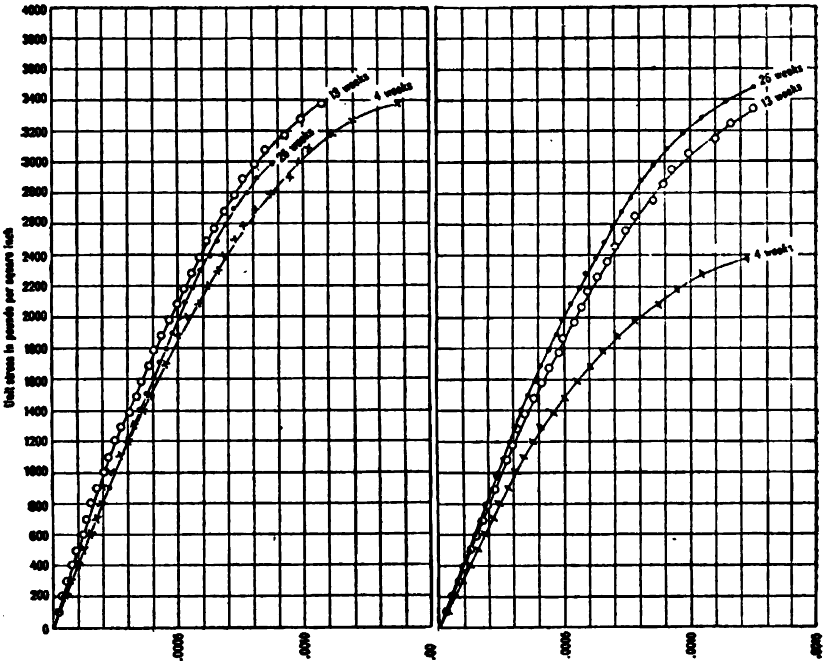

Fig. 34.

Deformation Per Unit of Length
Characteristic compression-stress deformation diagrams, granite concrete of medium consistency; ages 4, 13, and 26 weeks.

Deformation Per Unit of Length
Characteristic compression-stress deformation diagrams, limestone concrete of medium consistency; ages 4, 13, and 26 weeks.

The phenomenon of static fatigue of resistance under constant intensity of loading at such low values as half or less than half the ultimate strength of the material thus appears as a phenomenon closely related to the relief of primitive or preinduced residual strains in the specimen.

Concrete, because of the nature of its curing and its shrinkage in hardening, is a material in which internal residual strain of more or less intensity is invariably present and as a result of this condition apparent static fatigue of resistance is manifested during any continued initial application of constant compressive force at even relatively low intensities*. With the continued aging and curing of the specimen a marked change of its resistance occurs as is clearly illustrated by Fig. 34, which presents the stress strain curve of specimens from Bulletin No. 344 U. S. Bureau of Standards.

True static fatigue of a lean mixture of concrete in a floor structure has been discussed, Art. 6, Chapter 1, and such mixtures should be avoided. The most economical kind of concrete in reinforced work is a rich concrete which approaches true elasticity and which while it may exhibit apparent fatigue, like steel, is free from elastic fatigue under ranges of stress which do not exceed the intensity of reasonable and safe working stress.

Static fatigue of the direct tensile resistance of concrete in beams is illustrated by the endurance test published by †Hatt, Proc. Am. Soc. Testing Materials, 1907, page 425:

Figured Steel Stress of	Beam No.	Deflections April 11	May 3	Sept. 18
3000	22	.04	.08	.12
3000	23	.04	.07	.15
8000	26	.10	.13	.25
8000	21	.11	.11	.18

It will be noted that where the figured stress was but 3000 pounds per inch for a ratio of $f_s/f_c = 15$ there would be 200 pounds per inch on the concrete under common assumptions. For 8000 pounds on the same with the same ratio the stress in parallel on the concrete would be approximately 600 pounds. Even under the low stress of 200 pounds, the increase in deflection was 200 percent between April 11 and September 18.

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*See article by Moore & Fuller Proceedings of the Am. Conc. Inst. Vol. XII, 1916.

†The scientific accuracy of tests by Hatt above quoted could not be accepted by the authors had they not independently confirmed them, particularly in the light of the doubt cast upon them in Hatt's commercial testimony that there was but a slight increase in these deflections as disclosed by experiments noted. (Hatt RDQ 325 Turner v. Lauter Rec., Third C. C. A.)

These experiments verified by less extensive investigation by the authors, fully justify the total disregard of the direct tensile resistance of concrete by the practical builder because it so completely vanishes in a relatively short period in the time stress curve determined by these observations.

Change in configuration indicating *apparent* but not true static fatigue occurs not only in independent specimens but to a more marked degree in monolithic reinforced concrete columns and floor construction. Unlike a mere sample, such structures have induced in them both elastic and residual strains from the shrinkage and flow of the concrete into place which are localized to a considerable intensity at certain points.

It is customary to cast the columns and floor continuously, to secure the advantages of a true monolith. After pouring a column the excess water in the mix leaks away from the forms accompanied by the settlement of the semi-plastic mass in the column shaft and the drawing of the semi plastic mass downward from the floor or slab toward and into the column shaft or cap. Such action produces tensile shrinkage stresses of considerable intensity in the immediate vicinity of the junction points and the relief of these strains in the floor or beam under initial loading becomes quite apparent if measured by the extensometer.

Such relief of residual shrinkage strains does not represent elastic resistance nor is it related thereto. Such settlement of the semi-plastic matrix in the column shaft under the conditions noted gives rise not only to residual shrinkage strains at the underside of the floor next to the junction of the column but actual cracks in the top take place before the matrix has acquired much more than a semi-solid consistency. The concrete being in its original form, cement, sand and mud, as the water dries out it presents characteristics comparable with that of the mud in a slough bottom drying out, cracking and opening up. Such cracks, however, as those that occur about the column are usually filled with the drip from the succeeding or following floor and are rarely discovered by a research student until after a test load is applied to the floor. When thus discovered they are apt to be incorrectly regarded as cracks brought about by the application of the load rather than as cracks that existed before its application.

The value of published extensometer tests of flat slab floors resides in their interpretation and use in checking sound theory. To make interpretation of all readings as tho they were the result of elastic phenomena alone without any attempt to separate elastic from inelastic deformations or from the relief of residual strains, as may be readily done by noting the elastic recovery on removal of the load, is an error that has largely vitiated the conclusiveness of published deductions from such tests. Such omissions and oversights render reported extensometer measurements of initial loading of concrete floors lacking as scientific indications of all the concrete stresses. They have some value, however, in the general average of steel deformations although in the case of individual measurements on bars which have been bent cold they are without scientific value unless the elastic recovery on removal of the load is noted and deductions made therefrom instead of from the initial gage readings.

In the light of known existence of residual strains from shrinkage and the known nature of these strains, the error of undertaking to interpret extensometer measurements of deformations produced in the course of a week or ten days in concrete specimens under test load without allowing for the time effect of apparent static fatigue through more than the short space of forty minutes or an hour as against a week in a test structure, is too illogical to require criticism further than calling pointed attention thereto.

The magnitude of the error involved when the localization of shrinkage stresses in the practical structure is not considered frequently amounts to 35 to 40 percent of the amount of the erroneously interpreted concrete stress. The concentration of residual shrinkage strain and its relief under initial loading causes a corresponding error in all steel stresses measured immediately above the edge of the column cap. A short bend in the steel at the top is produced under load by the relief of shrinkage strain in the concrete at the bottom of the slab at the edge of the cap and at times an unbalancing of residual strain induced by cold bending occurs when the steel is bent at that point. The error from this cause, like that in the measurement of the deformation of the concrete, can be eliminated by check measurement of the elastic recovery on removal of the load, thus determining its amount.

13. Concrete Under Repeated Loading. We have shown in the discussion of residual strains that materials may be gradually relieved from their injurious effect by change of temperature and likewise by applied force changing the form repeatedly, if the force does not exceed certain well defined limits. As in the case of steel specimens, the elimination of residual strain occurs under fluctuating load stresses so in concrete specimens applications of moderate loads cause the material to deport itself more and more nearly in accord with the laws of perfect elasticity by the elimination of residual shrinkage strains.

When the load is applied gradually, the shortening of the specimen which is at first small, increases more and more rapidly as the load increases as shown in the concave curve OY Fig 35, plotted with the applied loads as ordinates and the deformation as abscissas. As the loading is gradually removed the curve YO' takes a convex form and shows a permanent set OO' on a horizontal axis. On again applying the same load, the new stress strain curve starting from the new origin O' is still of concave form looked at from the right just as the original curve OY was, but to a lesser degree and for the same load as at Y the point Y' shows a smaller relative set than the set OO'. On unloading again the origin is moved slightly to O''. With several successive applications and removals of the same load, the origin is continually removed slightly to the right, but the movement becomes less and less until there is no additional permanent set. The permanent set of concrete appears then to be in a great measure reached under the first loading and for

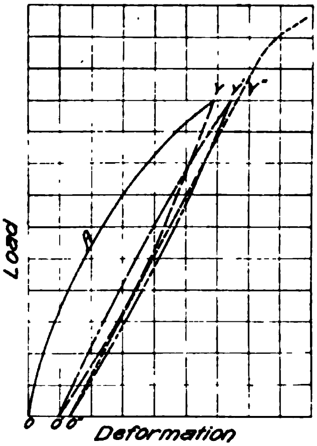


Fig. 35. Stress Strain Curves in Compression of Concrete under Repeated Loading.

subsequent applications of the same load the concrete acts more and more nearly as a perfectly elastic material.

The increase of permanent set after initial loading not only becomes rapidly smaller with increased number of repetitions of the applied load but it decreases greatly with the increase in hardness attendent upon the increase in age of the specimen. The fact that repeated loading of a rich concrete such as is required for reinforced concrete structural work causes the material to deport itself more and more nearly in accord with the laws of perfect elasticity renders its deviation from true elastic deportment in initial stages just after curing quite unimportant as the small deformations which relieve the residual shrinkage strains do not injuriously affect a practical structure properly proportioned and after these initial deformations have taken place under load the fact that no further or at least negligible deformations continue to take place renders the life of the concrete building under proper working stresses without known limit from the standpoint of deterioration and service.

14. Direct Tensile Stress in Concrete Across Reinforcing Bars Combined with the Shrinkage Stresses About Them. Shrinkage strains may be either elastic or residual. Elastic shrinkage strains which alone are here considered have reactions and properties in any material the opposite of residual strains as was shown previously in the case of bent steel bars.

The effect of cross embedment in flexure is not merely to reduce the section of the concrete normal to the bar, but also to weaken the ultimate direct tensile resistance of the concrete developed during the hardening or curing of the concrete by the amount of the elastic shrinkage strain about the bar. The simple experiments noted, Chapter I, last part of Article 9, show that the adhesion proper of the concrete in a direction normal to the surface of the steel is practically nil, and that the rigidity of the embedment along the bar depends upon tensile stresses developed in the concrete by the shrinkage of the concrete around the bar in curing. The longitudinal tensile resistance of the matrix across the bar is accordingly reduced by the full amount of these stresses acting upon a diametral plane thru the bar.

To investigate experimentally the effect of cross embedment, four 5″×12″ slabs of concrete, six feet long, were made, two plain and two reinforced crosswise with ½″ bars, six inches centers, ½″ from the bottom. See Fig. 36.

The calculated reduction in the moment of inertia of the section by reason of the ½″ bar is approximately two tenths, while the reduction in the modulus of rupture in the average of two tests was about five tenths below that of the plain concrete slab of the same mix and age, cured under the same conditions and tested in the same manner.

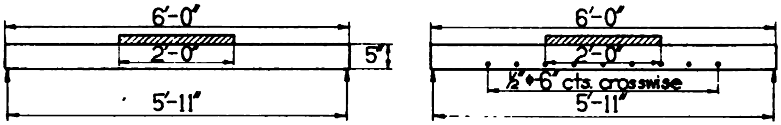


Fig. 36.

Now in the original Mushroom type slabs near the column there are areas of two, three and four layers of bars, one, two or three of which reduce the section and resistance of the concrete matrix in any given direction.

To investigate this effect still further experiments were made upon concrete slabs or wide beams like those already described, and also upon other slabs of the same dimensions, but having a cross embedment of rods at an angle of 45° with each other and at 67½° with the axis of the beam. Because of this obliquity of the reinforcing rods the moment of inertia of the slab was probably not made less but its modulus of rupture appears to have been reduced by the rods to a value between 25 and 35% of that of the plain slab.

From the foregoing, the attempt in the Proc. Am. Soc. C. E., January 1918, p. 246, to attribute the enormously greater carrying capacity of column supported flat slabs to a more persistent operation of the direct tensile stress in the concrete than can occur in beams appears futile. Since the cross embedment in slabs (which is generally absent in beams) reduces the modulus of rupture in beam action by the percentage above demonstrated, were the action of the column supported slab in fact similar to beam action we could readily double its strength by leaving the steel out entirely, since the latter on beam theory accounts for but twenty to twenty five percent of the applied moment. (See Table Chapter II, Art. 12).

The erroneous nature of the premise from which this conclusion is unavoidably reached will be further evident from a consideration of the phenomena of rupture by direct stress combined with temperature strains as follows:

15. Temperature Strains Combined With Direct Tensile Resistance of Concrete in Flexure. There are sound theoretical considerations as well as practical reasons, heretofore adduced, which justify building code regulations requiring that the direct tensile resistance of the concrete be entirely ignored in determining the strength of reinforced concrete floors.

Actual fracture of concrete floors in buildings occasionally takes place from temperature stresses, combined with working strain, particularly where there is an abrupt change in the cross section such as occurs where an elevator shaft cuts through a floor and largely reduces the section. A ground floor cast on walls having a rigid foundation on rock, is comparatively unyielding and checks under these conditions frequently occur. The case of floors higher up in a building, where the columns and walls may bend in and out, accommodating themselves to the expansion and contraction of the floor structure, presents more favorable conditions and corresponding freedom from temperature checks. The fact that ordinary floor slabs checked from temperature or shrinkage stresses continue to carry their working load and demonstrate their strength under test of more than double the working load is such a well known fact that the surprising degree of resistance of floor structures of Types III and IV can be accounted for only on the ground of a difference in mode of operation mechanically from mere beam action.

The coefficient of expansion of concrete is .0000065. If the concrete floor is cast and cured at a given uniform temperature and if the modulus of elasticity of the concrete in tension is 2,000,000, when a drop of 25° in temperature occurs, assuming the supports of the floor to be immovable, a temperature stress in the concrete equal to 2,000,000×25×.0000065, or 325 pounds per square inch will be developed, which approaches the ultimate resistance of plain concrete in tension.

It thus appears that if the floor were rigidly restrained at the ends, a drop of 25° below the temperature of curing might cause cracking of the concrete, unless inelastic elongation occurs with corresponding reduction in the value of E figured upon.

An experiment made on a slab 21′-0″ square 5″ thick well illustrates this point. The slab was supported on four sides by walls, and by a pier in the middle. Strips of poultry netting 24″ wide were run in four directions across the pier from wall to wall, but this section of metal was insufficient to increase the strength of the slab materially over that of plain concrete. The slab was loaded with concrete barrels, and withstood the load without sign of distress when first applied. The barrels were filled two thirds full of water and the total weight was 7½ tons. A drop in the temperature of 23° caused the complete collapse of the slab under this load.

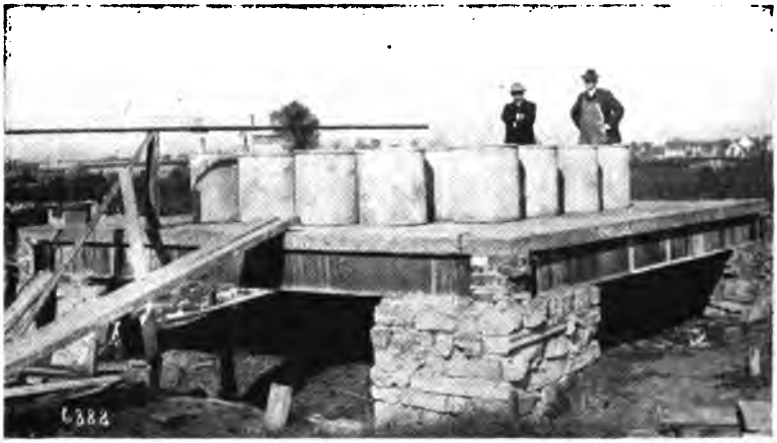


Fig. 38. VIEW

Load when first applied carried largely by tensile strength of concrete without distress.

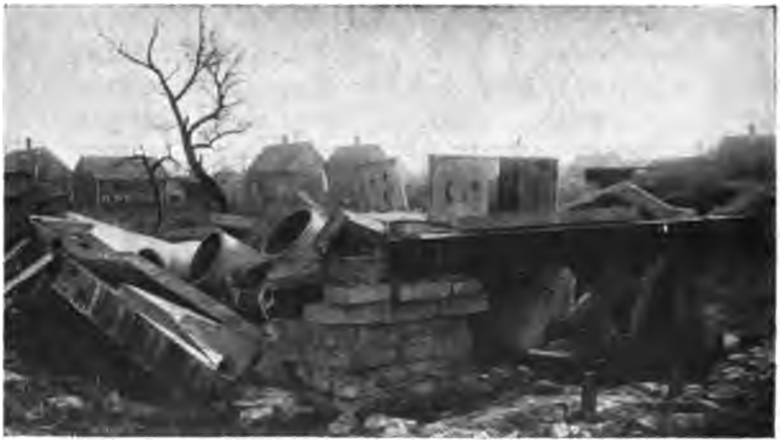
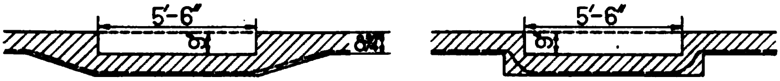


Fig. 39. VIEW

After two weeks failure of tensile strength of concrete occurred due to drop of 23° Fahr. in temperature.

The error of the supposition that the direct tensile strength of concrete plays any considerable part in the flexural resistance of practical flat slab floors will be evident on consideration of the fact that a similar slab well reinforced would carry without failure a load fifty times greater than that under which this collapse occurred.

How little in fact the tensile resistance of the concrete has to do with the performance of Type IV slabs may be illustrated by a mishap which occurred in putting up a Milwaukee building some ten years ago. A scale pit was to be sunk in the floor slab the plans requiring the forming as in Fig. 37.



As required by drawings. Fig. 37 As built.

The change as shown was made by the owners' inspector without consulting the Engineer. A full panel test of 600 pounds per sq. ft. twice the live load, was then applied next to the pit. As the last of the load was applied the rods kinked down at the side of the pit, pulled straight letting the slab bag six inches, but still carrying the hundred ton test load. The owner's attention was called to the fact that his inspector had, without permission, altered the design and he agreed immediately to pay the expense of repairs if the balance of the floor tested out. The sand used for the test was shifted to a similar panel the opposite side of the building but adjacent to the damaged panel and that test of course was satisfactory. No supports were placed under the damaged panel as it was of ample strength to support the test load through the bond shear resistance to sliding of the steel within the separated segments. Some of these segments were separated a quarter inch at the bottom but were squeezed together at the top thus resisting circular and radial compressions very effectively. About a third only of the resistance could be accounted for by tension in the steel operating by catenary action, hence the three elements of resistance were:

1. Compressive resistance radially and circularly of the cracked segments against each other.
2. The tension in the steel.
3. The bond about the steel preventing it from sliding laterally in the cracked segments.

The impossibility of utilizing the direct tensile strength of the concrete, even in beams with no cross embedment, is thus set forth in Reinforced Concrete by Turneaure and Maurer, 1911:

> "42 Tensile Strength and Elongation of Concrete when Reinforced:—We have seen that plain concrete has an ultimate tensile strength of about 200 lbs/in^2 and a total elongation of perhaps 1/1000 part, corresponding to a value of 2,000,000 for E_c. Steel stretches this amount under a stress of 30,000,000/1000 = 3000 lbs/in^2. Again the safe working tensile stress of concrete is about 50 lbs/in^2 and if we use a value of $E/E_c = 15$, the corresponding stress in the steel will be but 750 lbs/in^2. From these relations it is evident that in reinforced tension members we must either use very low and uneconomical working stresses for steel, or else expect the concrete to be of no assistance in carrying stress."

Even low stresses such as 50 lbs. sq. in. on the concrete are safe only in isolated masses such as footings, which could not be injuriously affected by temperature stresses.

Where the ratio of thickness to span length is sufficiently great and end restraint is present permitting arch action, safe construction is possible but not economical or commercially feasible because of the excessive thickness required. Stability under these conditions is secured by arched compressive stresses in the concrete, the direct tensile resistance playing no essential part in the operation of supporting the load under varying temperatures.

16. True and Nominal Factor of Safety. A popular misconception regarding the meaning of the term factor of safety as applied to steel construction has exerted an influence from the economic standpoint adverse to the rapid introduction of concrete construction.

Many have the mistaken idea that the factor of safety of four in steel construction means that the construction may be safely loaded to four times the rated working capacity; but this is not the case, since the yield point of steel is only about twice the working load; hence the actual factor of safety is practically two against the nominal factor of four.

In other words, the nominal factor of safety of four in structural steel work is based on the ultimate carrying strength in tension of the metal which is about four times the working load, but after the load has reached a little more than double the working load the yield point value of the steel has been nearly or quite reached and it commences to stretch, pulling out in case of mild steel before breaking sometimes as much as twenty percent or more of its total length. Evidently when this plastic distortion commences in a beam or column the member is soon so deformed that we cannot figure its strength in the frame, thus limiting the ultimate strength to practically a little more than twice the working load for this nominal factor of four.

In properly designed concrete construction the concrete is made stronger than the steel, for one reason because it is generally economical so to do, and hence the strength of the steel is the strength of the reinforced concrete construction and it would not be reasonable to expect to subject the steel to higher stresses in the case of concrete construction than is permissible in structural work; hence twice the working load is a fair test for this type of work. In reality, in view of the fact that the cement improves with age, if it will stand this test at an age of from three to four months the owner can rest assured that the factor of safety is greater than with structural steel construction.

Excessive test loads are not to be recommended since permanent strain is induced thereby and weakening of the structure may result therefrom. In case a test load of one and three quarters times the working load shows results in practical accord with elastic theory no greater loading is needed. The yield point of the steel in a structure is a critical point that determines what may be regarded as the practical ultimate strength of that structure. The nominal factor of safety is not in any wise related to the ultimate strength of the structure and is equally unrelated to the true limits of elasticity of the material; in fact it is nothing more than an irrational factor of ignorance originating before the advent of scientific analysis of construction. Before the statical analysis of the frame was developed the primitive idea of ultimate strength was that of a simple carrying rod and the idea was that the working strength should be a certain ratio to the ultimate tenacity of the material and this ratio was called the factor of safety which ratio is wholly inapplicable to a frame structure in which the elongation to develop the tenacity of the tension member would destroy the triangulation of the frame work. The advanced thinker in the engineering field today appreciates this so thoroughly that he avoids this term in his specifications and technical papers.

17. Working Stresses. Safe and practical working stresses must not exceed such limiting values that the working stress under loading repeated many times will not produce progressive change in the configuration of the structure on the one hand, nor a progressive accumulation of residual strains in any of its parts. Therefore, the working stress must be within the true limits of elasticity of the material by such a margin as will reasonably provide for additional subsidiary actions disregarded in the theory of predominant action from possible over-load and the effect of flaws in the material as well as the temperature strains that are incident to its configuration and exposure.

If we take the working stress as six tenths of the limit of elasticity of the material, that leaves a reasonable margin to provide for the above mentioned contingencies. The true elastic limit of soft steel (ultimate strength 55,000 to 60,000) may be taken at about 22,000 pounds per square inch. For steel of structural grade, (65,000 to 75,000 pounds) the elastic limit is about 26,000 pounds per square inch, and that of hard grade steel is about 30,000 pounds per square inch, permitting the following safe working stresses:

13,000 pounds per square inch for soft steel,
16,000 pounds per square inch for structural grade,
18,000 pounds per square inch for hard grade steel.

Concrete in Compression. The limit of elasticity in compression of a rich concrete such as a mixture of one cement to two sand, with four aggregate, or richer, may be taken at approximately four tenths of the ultimate compressive strength. In a leaner mixture, such as a 1 cement, to 3 sand to 4½ or 5 aggregagte, the approxmate limit of elasticity is in the neighborhood of 35 percent of the ultimate compressive strength, while in a still leaner mortar of one cement to 3½ sand the limit of elasticity is more nearly 25 per cent of the ultimate strength.

(Mix)	Elastic Limit	Working stress
1 : 2 : 4	1200 lbs.	720 lbs. sq. in.
1 : 3 : 5	760 lbs.	420 lbs. sq. in.
1 : 3½ : 7	450 lbs.	270 lbs. sq. in.

Wherever in practice the limit of elasticity of the concrete is exceeded, progressive static fatigue occurs under constant stress or load. Repeated stresses then tend to induce residual strain and ultimately wear out the material.

From the above relations it is evident that a rich concrete is the cheaper and more practical material to use, and less of it can be employed with added certainty. Since the rich mixture is more dependable and varies less in its ultimate strength and is more uniform, tough and dependable it is obviously the grade of material most suitable for use in reinforced concrete construction.

Working stresses for concrete in tension are to be treated on the same basis. The ultimate strength of a good 1 : 3 : 5 concrete in tension is probably in the neighborhood of 250 to 300 pounds.

Thirty five percent of this would bring the elastic limit in tension down to approximately 100 pounds per square inch and six tenths of that will be about 60 pounds. But 60 pounds per square inch in tension is so much below the ordinary temperature stress to which the concrete is subjected in the building that no safe working stress can be imposed upon the concrete in tension except in the case of independent footings where temperature strains are quite small. There 40 to 50 pounds per square inch on the concrete will give dependable results.

18. Concluding Remarks. From the foregoing discussion of elastic and plastic phenomena of deformation the following conclusions may be drawn:

Fluctuating stresses imposed without shock upon material within the limits of elasticity tend to render the material more perfectly elastic, tougher and more resilient. Stresses beyond the true limits of elasticity produce the opposite effect, cause brittleness, and by accumulation of residual strains may ultimately produce rupture at a stress below the ultimate resistance of the material under a load gradually increased and continuously applied.

The ultimate strength in compression of a sample of concrete because of hysteresis or lag in the progress of deformation becomes greater the more rapidly the load is applied in the testing machine and decreases considerably with a sufficiently slow application. Also, after a few repetitions of load within the limits of elasticity a specimen retains a nearly constant configuration with continuance of load, but beyond this limit there is a time curve of deformation which shows how the progressive shortening of the material under compressive stress continues at a less and less rapid rate until after a longer or shorter period of time the curve becomes asymptotic.

In order to make a scientific comparison of the results of different tests of the ultimate compressive strength of concrete it is not only necessary to adopt some standard form and dimensions for test specimens but also a prescribed uniformity of speed in applying the load.

In extensometer tests in which the strain is carried beyond the limits of nearly perfect elasticity of the concrete, allowance should be made for the time stress curve, preferably by noting gage point readings after removal of the load since by this means correction for effect of localized shrinkage stress is also effected.

Another question presented by the preceding discussion is how sharp a bend may a reinforcing rod receive without undue injury in the body of the bar as a tension element.

This question involves a double complexity—the effect of a sharp bend on the concrete in accentuating the strains therein and the damage done the bar itself by too sharp bending. A radius of bend 6½ to 7 inches should preferably be adopted rather than the sharp bend commonly made with many bending machines now on the market. Bends of such easy curvature avoid concentration of strain upon the matrix and prevent excessive residual strain in hard grade steel of good quality up to 1″ or 1⅛″ diameter. When the bars are larger than this it is preferable to increase the radius.

CHAPTER V.

THE CHEMICAL AND MECHANICAL ENERGY OF HARDENING OF PORTLAND CEMENT CONCRETE AND WORK PERFORMED BY IT AND RELATED PHENOMENA.

1. The hardening of concrete is a chemical action. No chemical change takes place without either liberation or absorption of heat and the rapidity of the change is in some measure dependent on the temperature at which it occurs. At very low temperatures all chemical action ceases.

Thus the hardening of concrete proceeds rapidly when the materials are heated to begin with and kept warm or when the temperature when casting is that of summer, but it proceeds with extreme slowness when the temperature falls to about 40 degrees Fahr., and ceases entirely at slightly lower temperatures when the water with which the concrete was mixed was only a little above the freezing point and the sand and stone also cold. The first day or two after pouring the chemical action of hardening responds quickly to increase of external temperature. Where the concrete has been kept at a low temperature for a month, there is very little response to external increase of temperature unless it continues for a long time. Therefore, in winter work it is important that the materials be heated for the purpose of accelerating the hardening and that heat be applied for one or two days after casting since it is economical to keep the concrete from even getting chilled.

Mr. W. A. Hoyt, Engineering Record, January 18, 1913, published his determination of the daily increments of compressive strength in pounds per square inch of cylinders, of a 1-2-4 gravel concrete kept at different temperatures, as shown in the accompanying table:

DAILY INCREMENT OF STRENGTH IN POUNDS PER SQUARE INCH 1 : 2 : 4 CONCRETE

Mean Daily Temp.	Age of Concrete in Days													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
32......	0	0	0											
34......	12	12	10	7	5	4	3	3	2	2	2	2	2	1
36......	25	25	20	15	10	8	7	6	5	4	3	3	3	2
38.	37	37	30	22	15	12	10	9	7	6	4	4	4	3
40......	50	50	40	30	20	16	14	12	10	8	6	6	6	4
42......	62	62	50	37	25	20	17	15	12	10	7	7	7	5
44......	75	75	60	45	30	24	21	18	15	12	9	9	9	6
46......	87	87	70	52	35	28	24	21	17	14	10	10	10	7
48......	100	100	80	60	40	32	28	24	20	16	12	12	12	8
50......	112	112	90	67	45	36	31	27	22	18	13	13	13	9
52......	125	125	100	75	50	40	35	30	25	20	15	15	15	10
54......	137	137	110	82	55	44	38	33	27	22	16	16	16	11
56......	150	150	120	90	60	48	42	36	30	24	18	18	18	12
58......	162	162	130	97	65	52	45	39	32	26	19	19	19	13
60......	175	175	140	105	70	56	49	42	35	28	21	21	21	14
62......	187	187	150	112	75	60	52	45	38	30	22	22	22	15
64......	200	200	160	120	80	64	55	48	40	32	24	24	24	16
66......	212	212	170	127	85	68	60	51	42	34	25	25	25	17
68......	225	225	180	135	90	72	63	54	45	36	27	27	27	18
70......	237	237	190	142	95	76	67	57	47	38	28	28	28	19
72......	250	250	200	150	100	80	70	60	50	40	30	30	30	20

His results agree with the author's experience in years of winter work in casting thin reinforced concrete slabs in building construction. Economy demands the application of heat and its continuation during the early stages of curing. Otherwise, much needless expense is involved in heating the uncured floors since at too late a stage in the curing a given amount of heat has a relatively insignificant effect.

2. Cold Weather Work.

At all temperatures below 45 degrees Fahr., it is best to warm the water to wake the cement up, otherwise it is liable to set too slowly to enable the forms to be safely removed at the usual intervals common in warm weather.

The water may be heated to 160 or 180 degrees and the sand and stone mixed with water in the machine before adding the cement. The water will thus warm up the stone and when the cement is dumped into the mixer the temperature will probably be in the neighborhood of 120 degrees.

When the temperature is below zero, boiling hot water is sometimes used. The sand and stone are first placed in the mixer, then the boiling water is added to warm up the sand and stone, and finally the cement, when the sand and stone has been warmed up and the water has been cooled down to a temperature of not more then 120°. This method of proceedure is necessary in order to not than 120°. This method of procedure is necessary in order to not kill the cement by the boiling water.

At low temperatures salt may be advantageously used. The proportion of salt which it is desirable to use for temperatures between zero and 25 above zero is a pint and one-half of salt to each batch containing two bags of cement. Below zero, a little more than a pint per sack, and extra care is to be taken in heating the materials and seeing that the concrete gets into place *hot.*

For these very low temperatures it is much better to heat the stone and sand over a coil of steam pipe if such is available.

In large masses such as thick walls, thick slabs, and the like, the cement generates heat in setting, sufficient to keep the body of the material warm, whereas in thin slabs this is not always the case and the concrete may and frequently does freeze.

It comes then to a question as to how to handle the concrete in the best and most practical manner. In a building having exterior bearing walls the walls are built up, then the slab is cast and artificial heat should be promptly applied on the under side of the slab to sweat out the concrete and enable it to harden up promptly. Window openings may be readily closed with canvas or light cloth.

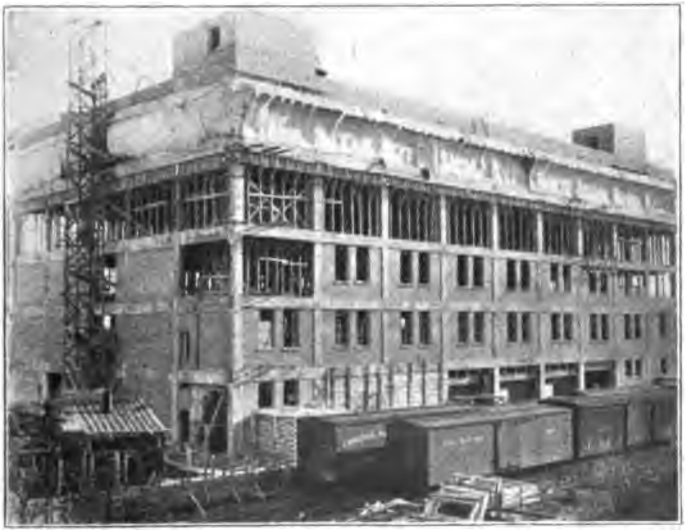


Fig. 40. Canvas curtain protecting an upper story of the International Harvester Building, Fort William, Ont. under construction in winter.

Where the building is a skeleton concrete frame it should be protected in winter from outside temperatures by help of canvas curtains in lieu of the walls as shown in Fig. 40. Then the concrete may be artifically heated and hardened.

In a large piece of work the most economical method of heating is to put in a small fan with the usual steam coil, and heat the building by blowing in heated air in the usual manner. In a small building this is too expensive and the resort is had to salamanders and coke for heating.

Attention is called pointedly to the necessity of melting snow and ice on old work and on forms before casting concrete and it remains to call attention to the necessity of special care in splicing the work.

The old concrete may be frozen and not hardened. It will be killed or disintegrated by heating with hot water as some thoughtless foremen have tried to do. Splices in the work should be made with great care and in a vertical plane both for beams and slabs. The old concrete should be cleaned of snow and ice with a steam hose, but no hot water used, then the new concrete may be cast against it and the moderate temperature of the new concrete will gradually soften the old work if frozen and the result will be a satisfactory bond between the two. It is preferable where practicable to continue casting until a whole floor is complete unless the work is of too great magnitude.

Inclined splices and irregular joints are very decided sources of weakness in work cast in cold weather, in fact, they can hardly be made good unless by digging out some of the concrete and thoroly grouting the joint after the work has hardened.

The foreman should be cautioned against killing cement by mixing with boiling hot water. Mixing the sand and stone first with boiling water will take the frost out of the stone and sand and warm it up and reduce the temperature of the mix down to 120 or 130 degrees which will not injure the cement. If there is ample salt this temperature may be even ten or fifteen degrees higher for a few minutes and not materially damage the mix.

We have known of cases where the concrete was placed in December, mixed with cold water, frozen as fast as placed, and when this same material thawed out in March it was as soft as the day when first cast. After the concrete had been kept thoroly wet for two and one-half weeks and then allowed to dry out, a good hard concrete was secured which after eight months stood an exceptionally satisfactory test.

Concrete unless promptly thawed out after it has been frozen, sets so slowly that its hardening may be condemned as altogether too slow for practical purposes if it is expected to clean up the work and get it finished within a reasonable time, and for this reason it pays the constructor well to heat the materials so that the centers may be removed promptly and the work finished up nearly, if not quite, as rapidly as it is ordinarily done in the summer.

Too great care, however, cannot be exercised in handling work during the winter season since frozen or partly frozen concrete may stand well when the forms are first removed and then as soon as it commences to thaw it will begin to get out of shape and look badly if it does not entirely collapse.

Many mistakes in judgment are made in handling work in the cold season of the year, although by the exercise of care and good judgment there is no reason why the work cannot be executed in a thoroly first class and satisfactory manner.

3. Caution Regarding Removal of Forms. A word of caution to the builder may not be amiss under this heading. The rapidity of the setting of concrete and hence the time at which it is safe to remove the forms varies materially, depending on the humidity of the atmosphere. In damp, rainy, wet, chilly weather, concrete is liable to set very slowly indeed. In dry weather, and particularly in high altitudes the concrete sets much more rapidly. In central Minnesota for example, under usual conditions, concrete may be counted upon to set more rapidly at temperatures ten or fifteen degrees lower than in situations close to the Great Lakes or in the South where the average humidity is much greater. The experienced superintendent soon becomes familiar with these conditions for a given locality, but if he moves about it is well to bear these general facts in mind since he will find a marked difference in different sections even with the same cement.

As to the time of removal of the forms, the builder should bear the fact clearly in mind that it is not the number of days time that the concrete has been in place or has stood upon the forms that determines whether it is safe to remove the forms, but the degree of hardness that has been attained during that period. Concrete may remain on the forms for four months in a northern climate, freeze and thaw out in spring and be as soft as the day on which it was placed, if the foreman has been so lacking in judgment as to use cold water to mix the concrete and then allow the material to freeze after it has been placed.

In the chilly weather of the spring or fall, if the concrete has been mixed with cold water and subsequently chilled by frost before it has had time to set appreciably it may be very slow in curing. It is exceedingly difficult under such circumstances for the most expert to tell when the false work may be removed and no undesirable results follow. The concrete may after such treatment, (improper mixing with cold water at a time when the weather is chilly and frosty) apparently be hard and subsequently sweat and soften during the continued hardening process and the work get out of shape. That is, the slab or beam may deflect permanently ½ or ¾″. Such deflection, while it will not result in permanent weakness after the concrete has finally hardened, will cause the owner to question its strength. It may cause partitions, which were built upon it at its original elevation, to be left unsupported and a year or a year and one half after the inelastic sag has occurred, the partitions will commence to crack and come down to a bearing. The owner will feel certain that the concrete work is weak, although it has hardened up and is of ample strength, tho not in the position in which it was left when the forms were removed. These troubles are entirely obviated by the proper treatment of the materials in mixing, as we have explained heretofore.

The danger of accident with half-hardened concrete is comparatively remote with multiple way systems, as this type of construction will almost invariably give the workman ample time to note its yielding and to prop it up before excessive deflection has occurred. Unfortunately, this is not the case with one-way reinforcement. When it once starts yielding as a rule the whole construction goes by the run, and hence from the standpoint of safety to workmen, the superintendent should exercise extreme care with this type of construction.

The question of determining the hardness of the concrete and whether it is safe to remove the forms is one which the builder must decide. As a rough test the concrete should be so hard that a twenty penny nail will double over and cannot be driven into it more than three-quarters of an inch. A good idea can be obtained as to its hardness by trying it with a hammer and seeing how readily it can be indented, as well as by driving a nail into it and finding out its condition under the surface. Examining the concrete around openings, etc., will enable the experienced foreman to form a correct judgment as to whether it is safe to remove the centering. In any case, these rough tests are sufficient to determine whether the removal of the forms is safe for the workmen.

In long span slabs or beams there may be some sag owing to compression of the concrete if it has not set sufficiently hard altho no accident may result. Such deflection or sagging tends to destroy the owner's confidence in the work altho it may have no material effect from the standpoint of strength. In fact, where a long span slab or beam has sagged a moderate amount before the concrete is thoroly hard there is generally little loss of strength.

In the case of a slab, if it is afterwards leveled up with additional concrete, it is stronger than that portion of the work which has kept its shape. The owner considers this an evidence of weakness. The builder knows that if he has filled up the depression in a panel which has sagged slightly it is probably the strongest slab in the building and a test of it will give exceptionally fine results.

The time during which the centering should remain in place varies for different spans. With a slab sixteen or seventeen feet square and seven inches thick under favorable drying conditions, it should be possible to remove the forms in eight or ten days. Where the span is longer, say twenty or twenty-five feet, two to three weeks at least should be allowed unless the slab is extra thick. For example, a slab eight inches in thickness and twenty-five feet in span should be allowed three weeks under the most favorable conditions to thoroly harden if it is to keep its shape immediately after removal of the supporting forms. Whereas a span of the same length, thirteen inches thick would only need practically the same time as the shorter span on account of the additional thickness. These are practical points which it is well to bear in mind as upon them commercial success in a measure depends.

The superintendent should bear in mind, in freezing weather, that concrete is as readily stiffened up by frost as by the true chemical action of hardening and that when thus hardened by frost it only remains for a rise in temperature to occur for the work to get out of shape, if it does not actually collapse. The test in freezing weather to determine whether the concrete has been merely stiffened up by cold or is actually cured, is to dig out a small sample and place it upon a hot stove, then if it sweats and softens the forms must remain in place. If, on the other hand, it remains hard and rigid and does not sweat and soften up then the concrete may be depended upon to retain its shape and forms may be safely removed.

4. Beneficial Effect of Heating the Water. If a drop of water be placed on a piece of cold glass, it will stand out upon it in the form of a globule instead of spreading out over the surface of the glass.

Heating the glass on the under side will cause the globule of water to flatten out and spread over the surface. This deportment of the water upon a cold surface is ascribed by chemists to "surface tension". Surface tension operates to prevent an intimate mixture of the water, cement and sand. It is greatly reduced by heating the water and for that reason where proper care has been exercised in this particular, concrete made in winter is of better quality than that made in the cool seasons of spring and autumn, and the materials not heated.

Whenever the temperature is below 45 degrees, it is both profitable and advantageous to use warm water in the mix because not only can the forms be removed earlier, but a better concrete results.

5. Shrinkage. The active initial forces producing shrinkage of the plastic concrete in the building are the force of gravity and the settling tendency caused by the leakage of the excess water from the plastic mix in the forms. Consequently there is a relatively large vertical shrinkage in a column and a much smaller vertical shrinkage in a beam or slab. The principal shrinkage in the column accordingly is lengthwise, the principal shrinkage in a horizontal beam or slab is crosswise of its length or vertically, with very little longitudinal shrinkage. The relative magnitude of these shrinkages in the plastic mix are roughly proportional to the height of the column cast at one time, and the depth or thickness of the slab, varying with the degree of fluidity or plasticity of the mixture. These primary conditions of shrinkage tend to produce elastic tensions in the horizontal steel reinforcement of the beams or slabs, and elastic compressions in the vertical steel of the column.

6. Work Performed on the Concrete and Metal by the Chemical Energy of Curing. The heat generated by the chemical action of setting is energy capable of doing work, the amount of which depends upon the intensity and duration of and manner in which the heat is expended or dissipated.

The heat generated by the setting of concrete is most noticeable in lining a tunnel. There, when it is 20 degrees below zero outside the men inside the tunnel are complelled to work with the least permissible clothing because of the heat thus generated by the concrete.

Mr. J. E. Howard, Tests of Metal, U. S. A., 1901, p. 130, found that the temperature of concrete setting in cubes was largely dependent on the size of the specimen, small cubes showing very slight increase since the radiation with the smaller cubes would be proportionately greater than with the larger ones. Volume for volume, the radiation is twice as great with a 6″ cube as with a 12″ cube, so that the heat generated is dissipated more rapidly by radiation from the smaller cubes and the increase in temperature is less. These relations must be considered in estimating the variation of temperature and the relative amount of work done on the materials by this chemical energy.

Under summer conditions a rise in temperature of 25 to 35 degrees in a 1–2–4 mix is probably a fair estimate for the interior of an ordinary concrete beam.

During the process of crystallization the chemical change under summer conditions goes on with considerable rapidity for two or three days, thus maintaining the heat within the body of the concrete during that period or longer, followed by a gradual reduction in temperature.

The steel, heated by the chemical action which is proceeding in the surrounding concrete, expands and is gripped in its elongated condition by the setting of the matrix. The concrete then loses heat by radiation from its exterior surface and these surfaces contract and harden while the concrete within the body of the member is still expanded by the higher temperature within the mass. When the interior of the concrete cools its free contraction is opposed by the exterior portion and it tends to set up shearing stresses between the exterior and the interior portions, and likewise between the steel and the concrete surrounding it. Expansion also accompanies crystallization and plays its part in the final result.

Thus it would appear that the steel tends to contract more than the concrete next to it, producing residual compressions in the concrete next to its surface, and elastic compressions beyond these immediate areas must balance the residual compression in the concrete and the tension thus induced in the steel.

The result of such a state of internal stress would, in the test of a beam, be represented by a knee in the lower portion of the steel deformation diagram, because the elastic compressions in the concrete, as they are gradually relieved by the application of the load permit the initial steel tensions and the residual compressions in the concrete to act. A similar knee occurs in the deflection curve while there is a relatively small change from a straight line in the compression curve of the upper fiber. This is well illustrated by typical diagrams of 4 and 52 week old specimens reproduced herewith from the Bureau of Standards Technological Paper No. 2, by Humphrey and Losse.

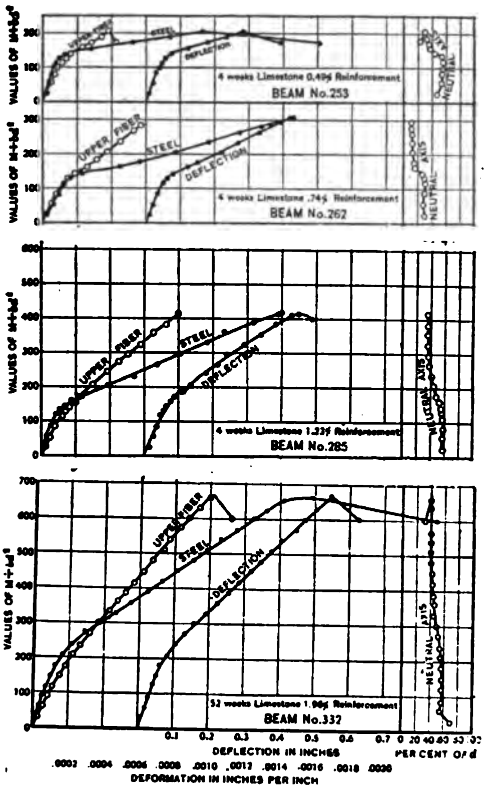


Fig. 41—Stress Deformation Curves of Limestone. Concrete at Different Ages with .49 and 1.96 per cent Reinforcement

It thus appears that the excess of rigidity in a newly cured beam over that due to the tensile resistance indicated by the measurements of the steel may be accounted for without presupposing a tensile resistance in the concrete which exceeds its ultimate tenacity and without the still more irrational assumption that as the concrete approaches its ultimate strength it can continually and regularly exert elastic resistance of a magnitude far beyond its true limits of elasticity.

Extensometer measurements of steel stresses determine the increments of stress shown by the measured elongations. In case the initial steel stress is unknown, as it is when it is produced by residual and elastic compressions in the concrete arising from the heat of curing and expansion due to crystallization the increment of the stress between the initial and the final stress is known but the final stress is not indicated by the extensometer reading and consequently is unknown since it can only be known when the initial stress is known; so that ordinary test measurements do not indicate total steel stresses since they give no indication of the initial tensions in the steel They only indicate increments or additions to the initial stress.

In summer, the heat of curing produces a state of stress of such intensity that the weight of the concrete is in a large measure removed from the supporting forms, a phenomenon which has never been observed in winter work, notwithstanding the fact that the atmosphere surrounding the forms is less humid than in summer by reason of the drying effect of artificial heat.

The conclusion that curing induces tension in the steel is the reverse of that arrived at by M. Considère based on experiments on the shrinkage of prisms 2.36" by 0.98" by 23.5 inches in length with small bars ¼ inch diameter embedded in tamped concrete, the properties of which differ from those of plastic poured concrete.* In such small sized specimens, Howard's tests mentioned above show that no material rise in temperature could occur since the heat would be radiated away so rapidly that it would have little effect upon the metal. The case is quite otherwise where there is a considerable body of concrete, for in that case a rise in temperature of 20 degrees would expand the steel as much as a stretching force of 3,900 pounds per square inch, and the results of test would therefore differ widely from those of the laboratory sample of Considère. Moreover, Considère used a relatively dry mix compacted by tamping instead of shrinking by loss of excess water.

Considère's experiments on concrete hardening in water show expansion which is partly duplicated in the wet mix and is followed by some contraction.

So far as known, no tests have heretofore been made to investigate the phenomenon just described.

In order to establish the facts as to this phenomenon, a beam 14" square by 14'0" long was cast in the City Testing Laboratory, Minneapolis† for the authors, early in December. The reinforcement consisted of two ⅝" round rods 3" from the bottom thruout. These rods extended thru the beams to about six inches beyond the ends and were bent back parallel to the side of the beam and cut 2" short of the center. These bars were supported upon cross rods ¾" in diameter 2'-0" on centers passing thru the beam laterally and supporting the longitudinal rods both inside and out, as per cuts, Figures 42 and 43. These cuts are made from photographs of the finished beam showing the thermometers for measurement of the temperature of the beam at *T*, the extensometer at *E*, placed in holes drilled in the rods for the measurement of changes of length. Two sets of holes had to be drilled in order to cover the large range in the change of length of the steel.

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*See Marsh, "Reinforced Concrete", 2nd Edition 1905, page 253.

†The assistance of Mr. P. Pearson, in charge of the Minneapolis Testing Department, in carrying out this test is acknowledged.

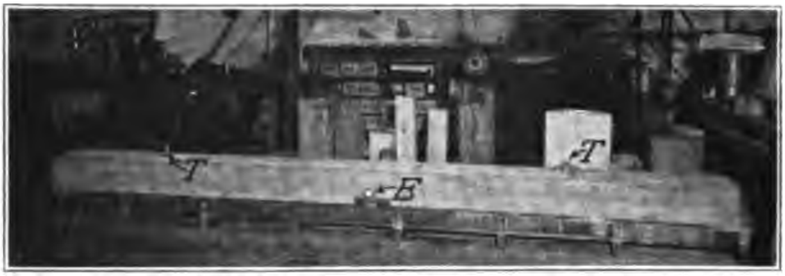


Fig. 42

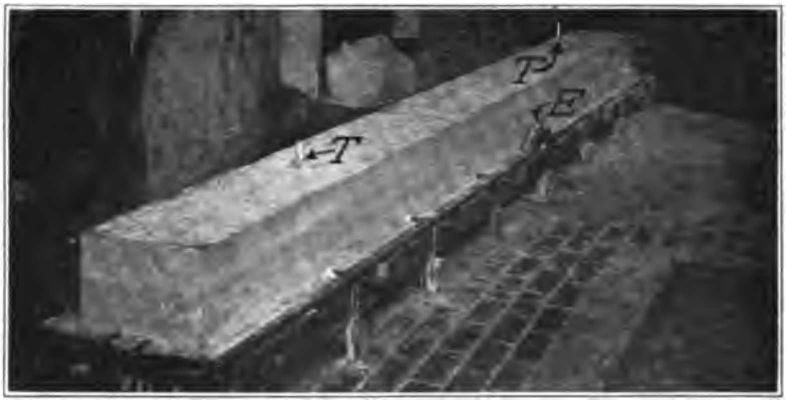


Fig. 43

The concrete was a standard 1 : 2 : 4 mixture, using washed gravel for the coarse aggregate. The temperature of the concrete, when poured, was 80° F., and that of the room 71° F. The temperature of the matrix, rose about four degrees and remained at this maximum for about 24 hours. The original temperature of the steel was 71° Fahr., five hours after the concrete was poured it had risen to 84°. This rise of 13° in temperature would elongate the steel .01326″. In five days the temperature of the concrete fell to 71°, the same as that of the room when it was poured, the temperature of the room at this time being 67°, and the average extensometer measurements of the elongation was .0449″; correcting this for four degrees lower temperature of the steel outside the form, we have an elongation of

.0408″. To produce this extension would require a stretching force of $(.0408/160) \times 3(10)^7 \doteq 7650$ pounds per square inch tension in the steel and this was the value of the induced stress in the steel at that time.

From this time contraction set in with the drying out and at the end of three weeks the average amount of extension was, by extensometer reading, .0427″, which corrected for two degrees drop of temperature inside the beam equals .0347″ or 6400 pounds per square inch tension in the steel would be required to produce this extension. The extension shown by the extensometer readings at the end of 40 days corresponded to 2145 pounds per square inch tension in the steel.

Expansion in the process of crystallization is dependent on the amount of water present. Thus, when the concrete is cured under water, a large expansion has been observed in the experiments of Bauschinger and those of Considère, as well. To the extent that the wet mix furnishes the excess water, expansion proceeds afterwards followed by shrinkage of the concrete as it dries out, corresponding to the observation of Considère in the shrinkage of dry tamped specimens when hardening in air.

In the summer the crystallization goes on rapidly before the water has a chance to dry out. In cold weather work where the hardening is much slower the excess water is largely eliminated and the expansion in the process of crystallization is greatly reduced. As the expansion from crystallization proceeds some readjustment takes place between the large tensile stresses in the steel and the stresses in the concrete about the steel, residual compression strains being developed in the concrete about the steel during the expansion in the course of crystallization.

Initial tension is to be expected in the steel in warm weather work and also where the concrete has been properly handled in cold weather. Where the concrete has been improperly taken care of in cold and chilly weather and for that reason has required several times the normal time to harden, the condition of internal stress remains to be determined. We may assume that the initial tensions and residual stresses are of lower intensity under these circumstances. Whether there is some initial compression in the concrete due to shrinkage under these abnormal conditions, is a matter yet to be investigated.

The radical difference between the curves with a small percentage of steel and the curves with a larger percentage of steel, that is with larger rods, exhibited in Fig. 41 from Paper No. 2 of the Bureau of Standards, may be accounted for as follows:

The bond resistance varies roughly with the area of the rod, for a given length of embedment. The cross section of the rod varies with the square of its diameter while the bond resistance per unit of length varies as its diameter. Consequently a rod 1″ diameter elongated the same amount as ½″ diameter would bring four times the bond stress divided by twice the area or two times the unit bond stress upon the concrete that the rod ½″ diameter would bring upon it. Consequently there is more slip with a larger rod than with a smaller rod or greater residual strains are produced in the concrete by the smaller size bars which do not slip. From the principles of equilibrium of residual and elastic strains it will be observed that both the initial elastic strain of tension in the steel and the residual compression strains in the concrete must be balanced and held in equilibrium in the compression zone by elastic compressions in the concrete.

From these results it is obvious that the prominent knee in the deflection and steel curves of beams under initial test is to be attributed principally to the factors above noted, and that the tensile resistance of concrete is a minor factor. That the knee in these curves becomes more, rather than less, pronounced with increase in age and strength of the specimen is indicated by the Bureau of Standards tests.

7. Hygrometric Condition of Concrete. It is well known that concrete expands with moisture and shrinks as it becomes dried out, resembling timber in this regard. The amount of this change is but a minute fraction of the changes in sidewalk slabs noted by Prof. A. H. White, Univ. of Mich., Engineering News July 6, 1911, July 9, 1914, since in a building the concrete is not exposed to the wetting and drying of sidewalk slabs.

The amount of residual moisture combining with the concrete after four weeks of curing weather is small but it has an appreciable effect upon the volume of the concrete in the compression zone as it dries out. The effect upon the tension zone which has been compressed by the residual tension induced by the heat of hardening seems to be neglible as evidenced by the same bend or knee in the deflection and steel stress curves of long cured test specimens.

This difference in the conditions of the tension and compression zones ultimately causes a slight sagging of every concrete floor or beam, the amount of which is inversely as the square of the ratio of the depth to span length for the same grade of concrete cured under identical conditions. In a design in which proper relations of depth to span are adhered to, and in which the concrete is of the mixture recommended, the amount of sag produced by the phenomena before discussed will be too small to cause difficulty.

8. **Mechanical Energy and its Effect on Hardening of Concrete.** We have shown that the chemical energy of heat in curing accelerates the hardening of Portland Cement concrete and at this point it is well to call attention to the fact that mechanical energy has the same effect provided the percentage of moisture is greatly reduced.

In the manufacture of the famous Spanish hydraulic tile, the moisture is contained in the facing of the block which is generally of neat cement and color, backed with dry cement and a mixture of dry cement and sand. Upon the application of the pressure, the moisture is disseminated through the backing and the tile can be handled and is as hard as an ordinary brick immediately after the application of pressure of 5000 pounds per square inch.

Summary. 9. The conclusions derived from the investigation of the effect of the work of chemical energy in curing reinforced concrete shows that book theories attributing the result of this work to an abnormal tensile resistance of the concrete, are largely erroneous. They have always been doubted by practical mechanics who have handled plain concrete and reinforced concrete slabs in building work just as practical mechanics in the bridge shop doubted the rigidity of the columns of the first ill-fated Quebec bridge design, because of their lack of rigidity when handled in the crane.

In initial tests of newly cured beams, the operation of the residual compressions and the elastic tension induced in the steel during curing have been mistakenly credited to the direct tensile resistance of the concrete itself.

CHAPTER VI.

MECHANICS OF COACTION OF METAL AND CONCRETE IN SIMPLE BEAM ACTION

1. When a load is placed upon a beam it tends to deflect or bend it. Reinforcing rods are brought into tension when situated in that part of a reinforced concrete beam which is strained in tension. There is then a tendency for the rods to slide longitudinally within the concrete matrix. This tendency to slide is resisted by the shrinkage grip which furnishes a shearing resistance that opposes longitudinal sliding. The nature of this shear may be illustrated by grasping a pencil horizontally in one hand and pulling on it with the other. The flesh will be observed to rotate in such a way as to produce compression at 45 degrees to the angle of pull on the pencil and to be drawn in toward the pencil at 45 degrees in the direction of the pull.

The effect of this shearing resistance upon the cooperation of the two parts of a body may be illustrated by considering a pile of four planks, of the same length with the ends placed even, under a load, and simply supported at the ends. As the planks bend under the load the lower corners of an upper plank slide past the upper corners of the lower plank just below it at the ends and when arranged in this way their combined stiffness and load capacity is the aggregate of that of the four planks.

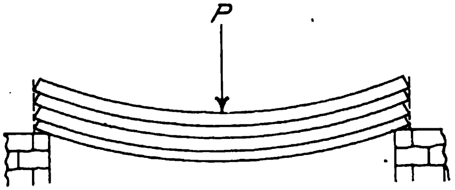


Fig. 44

Now instead of allowing the planks to slide upon each other, let us introduce horizontal shearing resistance between the planks by gluing them together so that they cannot slide one upon the other, then their combined strength is increased to four times that of the aggregate of the four planks operating independently. That is their strength in bending is equivalent to that of sixteen planks under the same load while the total stiffness is sixty four times that of a single plank. Thus the horizontal shear resistance furnished by the glue has greatly increased the stiffness and strength of the combination.

Likewise when concrete and metal are combined, the shrinkage grip or bond shear between them has an effect similar to that of the glue in the example of the planks above described which causes the steel and concrete to work together to produce a load carrying capacity far greater than that of the two elements not so combined.

Therefore, the three principal elements which acting together in the reinforced concrete beam determine its strength, are,

1. Compressive resistance of the concrete.
2. Tensile resistance of the steel.
3. Horizontal bond shear resistance furnished by the grip of the concrete on the steel which causes coaction of the concrete and steel thereby forming a combination which has many times the strength and stiffness of the two materials acting separately.

Because of the low value of the elastic limit of the concrete in tension and the fact that temperature stresses may readily exceed this limit, resistance to tension in the concrete cannot be counted upon as an element of resistance in a reinforced beam. The stresses across the section of the beam under elastic action which we can count upon with certainity are then the compressive stresses above the neutral axis and the tensile resistance of the steel below the neutral axis.

Notation adopted in discussion to follow:

1. Rectangular Beams.

f_s = Tensile unit stress in steel,
f_c = Compressive unit stress in concrete,
E_s = Modulus of elasticity of steel,
E_c = Modulus of elasticity of concrete,
$n = E_s/E_c$
e_s = elongation of steel due to f_s.
e_c = shortening of the concrete due to f_c,
T = Total tension in the steel at any section,
C = Total compression in the concrete at any section,
M = Moment of external force or moment in general,
Moment of Resistance = M_s for steel; M_c for concrete.

A = Steel area,
b = Breadth of beam,
d = Depth of beam to center of steel,
k = Ratio of depth of neutral axis to effective depth d,
z = Depth of resultant compression below top,
j = Ratio of lever arm of resisting couple to depth d,
$jd = d - z$ = Arm of resisting couple,
$p = A/bd$ Steel ratio (not percentage).

2. T-Beams.

b = Width of flange,
b' = Width of stem,
t = Thickness of flange.

3. Beams Reinforced for Compression.

A' = Area of compressive steel,
p' = Steel ratio for compressive steel,
f_s' = Compressive unit stress in steel,
C = Total compressive stress in concrete,
C' = Total compressive stress in steel,
d' = Depth to center of compressive steel,
z = Depth to resultant of C and C'.

4. Shear and Bond.

V = Total shear,
v = Shearing unit stress,
u = Bond stress per unit area of bar,
o = Circumference or perimeter of bar,
Σo = Sum of the perimeters of all bars.

2. Common Theory of Flexure Applied to Concrete: This theory is based on the assumption—

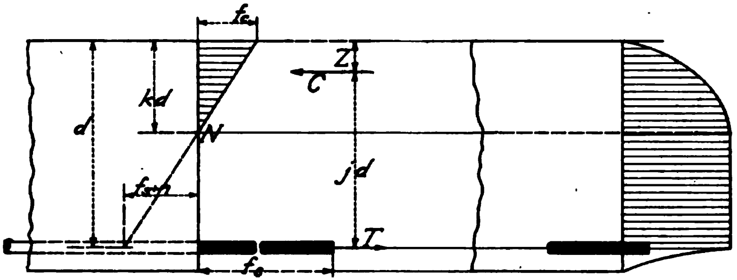
1. Any plane right cross section of an unloaded beam will still be plane after bending.

2. The material of the beam obeys Hooke's Law—stress is proportional to strain.

From the first assumption it follows that the unit deformations of the fibers at any right cross section of the beam are proportional to their distances from the neutral surface.

From the second assumption it follows that the unit stresses in the fibers at any section of the beam are proportional to the distances of the fibers from the neutral surface. This is called the linear law of distribution of stress, and while the assumptions are not perfectly correct in certain minor particulars to be discussed later, they are however, sufficiently accurate to serve the purpose of determining flange stresses.

This distribution of these stresses are shown graphically in figure 45.



Horizontal tensile and Compressive stresses Horizontal Shears
Fig. 45

3. Neutral Axis and Resisting Couple: From the linear law the unit deformation of the fibers varies directly as the distance from the neutral axis, or $e_s/e_c = (d-kd)/kd$. Also $e_s = f_s/E_s$, $e_c = f_c/E_c$. $E_s/E_c = n$.

$$\frac{f_s}{nf_c} = \frac{d-kd}{kd} = \frac{1-k}{k} \quad \text{.......................................} (1)$$

For equilibrium the total tension below the neutral axis equals the total compression above the neutral axis. Hence

$$f_sA = \tfrac{1}{2}f_cbkd \quad \text{.......................................} (2)$$

Eliminating f_s/f_c between equations (1) and (2) and introducing $p = A/bd$, there results

$$2pn\,(1-k) = k^2 \quad \text{.......................................} (3)$$

or $$k = \sqrt{2pn + (pn)^2} - pn \quad \text{.......................................} (4)$$

The equation shows that k increases as p or n increases and that for a given percentage of reinforcement the neutral axis in all beams is the same proportionate distance k from the top of the beam. The centroid of compressive stress is $\tfrac{1}{3}$ kd from the compression face of the beam. The arm of the resisting couple TC is

$$jd = d - \tfrac{1}{3}kd \text{ or } j = 1 - \tfrac{1}{3}k \quad \text{.......................................} (5)$$

As k increases j decreases, but in a different ratio.

Figure 46 shows how j changes with p for different values of n.

Diagram for Design of Double Reinforced Beams

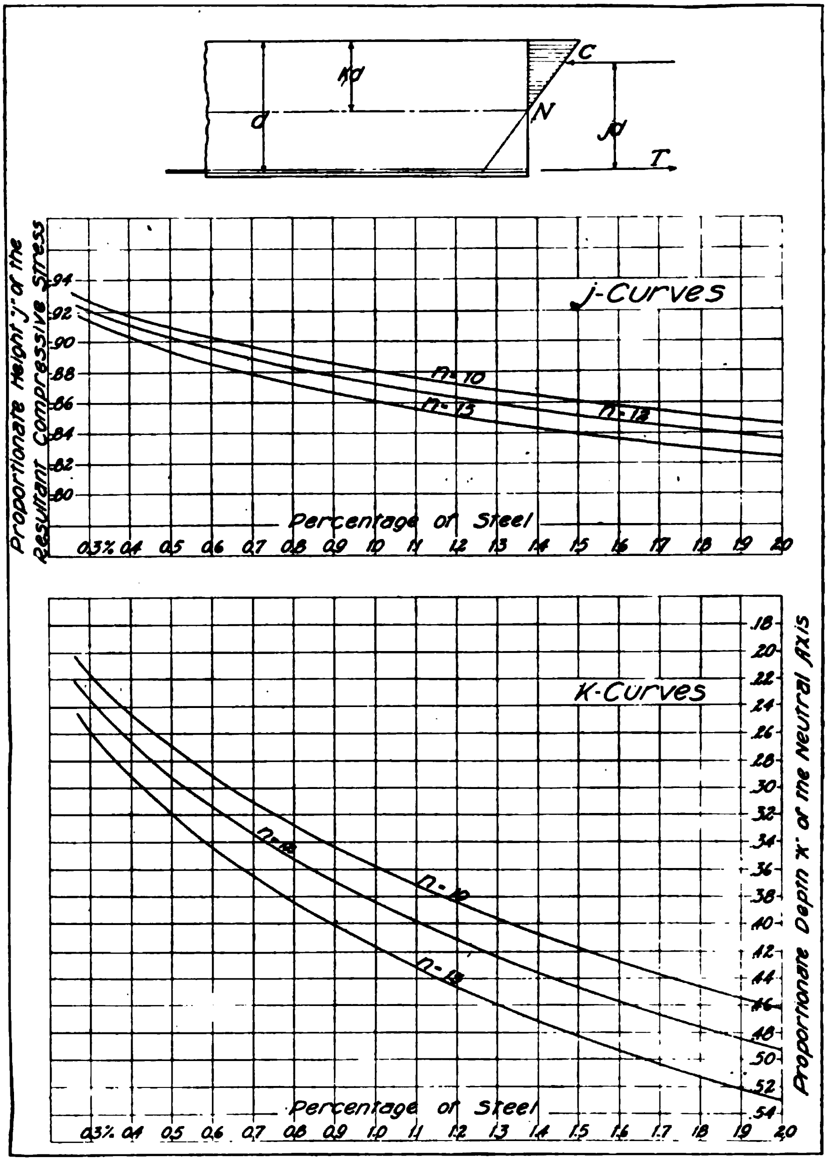


Fig. 46. j and k curves.

4. **Determining Moment.** In case the steel element in a reinforced concrete beam is weaker than the concrete, the determining resistance is that of the steel, and

$$M_s = Ajdf_s = pjbd^2f_s \qquad \text{(a)}$$

If, on the other hand, the beam be over reinforced, the determining moment is that of the concrete, and

$$M_c = \tfrac{1}{2}\, jkbd^2f_c \qquad \text{(b)}$$

For approximate calculations it will be sufficiently correct to assume $j=0.85$ and $k=0.40$, these being fair average values for steel percentages from 0.75 to 1.25. Equations (a) and (b) then become

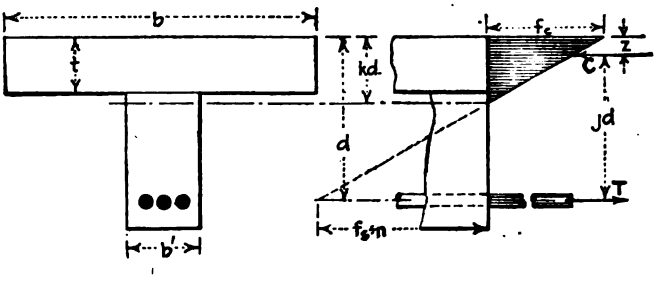
$$M_s = 0.85\, Adf_s$$
and $M_c = 0.17\, bd^2f_c$.

2. *T-Beams.*

Case I. When the neutral axis lies in the flange: Use the formulas for rectangular beams.

Case II. When the neutral axis lies in the stem.

The following formulas neglect the compression in the stem:



Position of neutral axis,

$$kd = \frac{2\, ndA + bt^2}{2\, nA + 2\, bt} \qquad \text{(6)}$$

Position of resultant compression,

$$z = \frac{3\, kd - 2t}{2\, kd - t} \cdot \frac{t}{3} \qquad \text{(7)}$$

Arm of resisting couple,

$$jd = d - z \qquad \text{(8)}$$

Fiber stresses,

$$f_s=\frac{M}{Ajd} \quad \dots\dots\dots\dots \quad (9)$$

$$f_c=\frac{Mkd}{bt\left(kd-\frac{1}{2}t\right)jd} \quad =\frac{f_s}{n}\cdot\frac{k}{1-k} \quad \dots\dots\dots\dots \quad (10)$$

(For approximate results, the formulas for rectangular beams may be used.)

The following formulas take into account the compression in the stem; they are recommended where the flange is small compared with the stem:

Position of neutral axis,

$$kd=\sqrt{\frac{2\,ndA+(b-b')t^2}{b'}+\left(\frac{nA+(b-b')t}{b'}\right)^2}-\frac{nA+(b-b')t}{b'} \quad (11)$$

Position of resultant compression,

$$z=\frac{\left(kdt^2-\frac{2}{3}t^2\right)b+\left[(kd-t)^2\left(t+\frac{1}{3}(kd-t)\right)\right]b'}{t(2kd-t)b+(kd-t)^2\,b'} \quad \dots\dots \quad (12)$$

Arm of resisting couple,

$$jd=d-z \quad \dots\dots\dots\dots \quad (13)$$

Fiber stresses,

$$f_s=\frac{M}{Ajd} \quad \dots\dots\dots\dots \quad (14)$$

$$f_c=\frac{2\,Mkd}{[(2\,kd-t)\,bt+(kd-t)^2\,b']\,jd} \quad \dots\dots\dots\dots \quad (15)$$

3. *Beams Reinforced for Compression.*

Position of neutral axis,

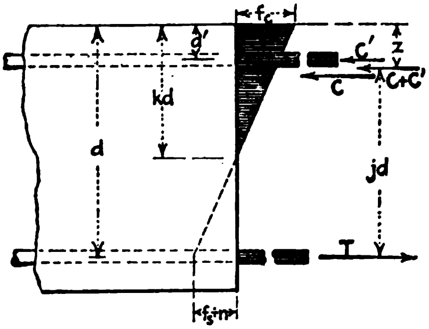
$$k=\sqrt{2\,n\,\left(p+p'\,d'/d\right)+n^2(p+p')^2}-n\,(p+p') \quad \dots\dots\dots\dots \quad (16)$$

Position of resultant compression,

$$z=\frac{\frac{1}{3}\,k^3d+2\,p'nd'(k-d'/d)}{k^2+2\,p'n\,(k-d'/d)} \quad \dots\dots\dots\dots \quad (17)$$

Arm of resisting couple,

$$jd = d - z \qquad (18)$$



Fiber stresses,

$$f_c = \frac{6M}{bd^2\left[3k - k^2 + \frac{6\,p'n}{k}(k - d'/d)\;(1 - d'/d)\right]} \qquad (19)$$

$$f_s = \frac{M}{pjbd^2} = nf_c\,\frac{1-k}{k} \qquad (20)$$

$$f'_s = nf_c\,(k - d'/d)/k \qquad (21)$$

The case of the doubly reinforced beam is one which the designer rarely is called upon to make use of. In it the compression steel should preferably be about 2½″ to 3″ from the top surface of the beam. Unless the percentage of tensile reinforcement is very high, say 3 percent or more, and the compressive reinforcement very low, say less than 0.75 percent, the neutral plane is nearer the compressive steel. Assuming d' to equal $d/10$ when p' is 2 percent, it is nearer the compressive steel for all values of p'. Thus it follows, since the unit stresses in the compressive and tensile reinforcements are as the distances of these reinforcements from the neutral plane, that the unit stress in the compressive steel is for these percentages less than that in the tensile steel. For very rough approximate computations, taking $n = 15$ and the average value of $j = .85$, $k = .45$, we have

$$f_s = \frac{1.17M}{p\,b\,d^2} \quad \text{and} \quad f_c = \frac{M}{(.19 - 10.5\,p')\,bd^2}$$

The above formula for f_s is a fair approximation. The formula for f_c with different percentages of steel is by no means a close approximation.

A much more satisfactory method of computation is as follows: From equations (20) we find

$$f_c = \frac{f_s k}{n(1-k)} \quad \text{(19a)}$$

To determine f_c we cannot assume an arbitrary value of k in this equation since that would be tantamount to assuming that the amount of compressive steel would make no difference in f_c, hence this equation cannot be used as an approximate method of determining f_c, but it may be employed to determine f_c in an accurate manner by plotting the curve from the values of $n(1-k)/k$ for different percentages of tensile and compressive steel from which we may derive f_c by dividing f_s by the value taken from the diagram.

The accompanying Fig. 47 shows the curves of different percentages of tensile steel for different percentages of compressive steel reinforcement noted at the bottom of this Fig. The values of j are given at the left of the diagram and the values of $f_s \div f_c$ for the different percentages of tensile and compressive reinforcing steel given at the right.

Diagram for Design of Double Reinforced Beams

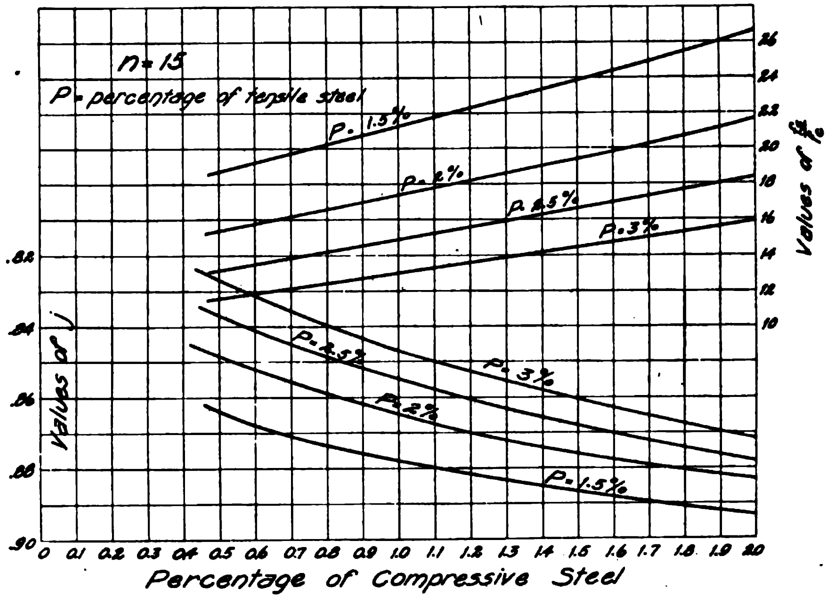


Fig. 47.

5. Bond Shear in Simply Supported Beams:

In the following formula, Σo refers only to the bars constituting the tension reinforcement at the section in question, and jd is the lever arm of the resisting couple at the section.

For rectangular beams,

$$v=\frac{V}{bjd} \quad \dots\dots\dots\dots\dots\dots\dots\dots (22)$$

$$u=\frac{V}{jd\,\Sigma o} \quad \dots\dots\dots\dots\dots\dots\dots\dots (23)$$

(For approximate results, j may be taken at $\frac{7}{8}$.)

The foregoing formula for bond is correct for simply supported beams only. The permissible bond shear stress is invariably greater than can be developed with proper working stress of tension upon the steel or compression in the concrete.

The Joint Committee suggest that it may be assumed that four percent of the compressive strength of the concrete for rolled rods but not more than two percent in the case of drawn wire and with the best type of deformed bar, five percent of the compressive strength of the concrete is a permissible value of the bond shear stress.

The common theory of flexure assumes that a plane right cross section still remains plane after bending and that it also lies in the direction of the radius of curvature of the beam. The fact is, however, that this section is twisted in a radial direction and also given a slight curvature by the horizontal shearing deformation. The error involved in both of these assumptions is radical in respect to the investigation of shear failure even in the simple beam and is much more so in the continuous beam. It is immaterial in determining the flange or fiber bending stresses and hence immaterial in the general design of the simple beams treated in this Chapter.

6. Tests of Simple Beams:

The accompanying figure illustrates tests of reinforced concrete T-beams at the University of Wisconsin. The customary manner of strengthening the web of the concrete by stirrups and bent up bars is here shown.

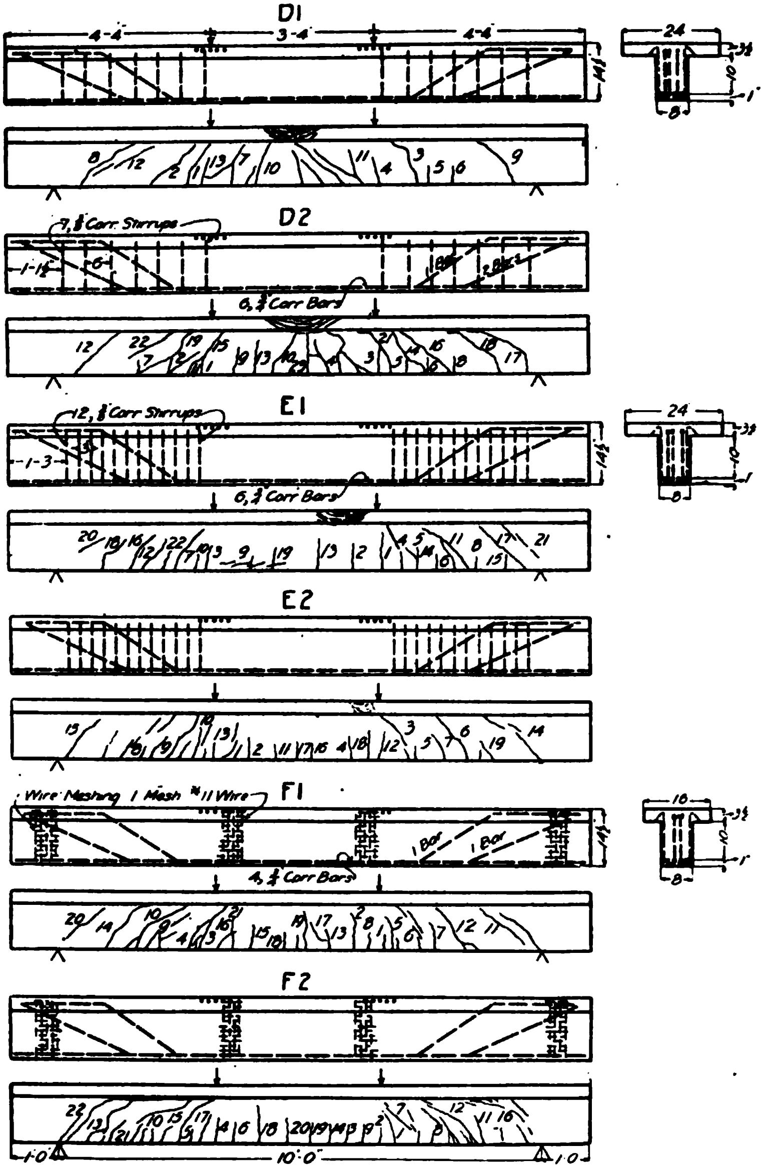


Fig. 48. Reinforcement of Test Beam and Cracks

When tested to destruction the concrete beam, simply supported, fails in one of three ways:—

(a) By yielding of the steel at or near the section of maximum bending.

(b) By crushing of the concrete at the center of the span near the top as outlined in the cut of Beam D1 in Figure 48.

(c) By diagonal shear or tension failure of the concrete at that place where the shearing stress reaches a certain limiting amount.

When the beam is progressively loaded until the steel reaches its yield point value further load by stretching the steel produces rapidly increased deflection, the opening up of large cracks in the concrete on the tension side followed by raising the neutral surface toward the compression side. This causes rapid increase in the compressive stress and ultimate failure by the crushing of the concrete. If the beam is relatively long and the percentage of steel high, the crushing strength of the concrete is reached before the yield point value of the steel. When failure by crushing of the concrete takes place the shear cracks in the concrete are smaller than where tension failure of the steel occurs.

Diagonal shear cracks occur whenever a certain limiting shearing deformation occurs in the beam. In case the beam is relatively long and shallow the curvature under uniform load would be sharpest toward the center and the horizontal shearing deformation greater than with a deeper beam under the same tensile stress in the steel. Consequently the concrete will crack toward the center. Where the load is applied at the third points of the beam as in the tests of the Bureau of Standards, Technological Paper No. 2, these cracks are approximately vertical but where the load is uniform they will be inclined toward the center because of the detrusion of the plane cross section from the plane of radius of curvature caused by horizontal shear.

7. Limiting Shear Values and Steel Ratios in Simply Supported Beams. In section 11, Chapter IV, the effect of sharpness of curvature in cracking the concrete has been discussed. Not only does the question of limiting shear stress in the concrete on the tension side of the beam in flexure require the adoption of small value of f_c for shallow beams but a desirable limit of deflection under working load requires that deflection shall not exceed $L/1000$. With f_s given, the adoption of large values of f_s/f_c will make the compressions in the concrete so moderate as to prevent excessive deflection even tho there should be some small initial deflection due to non-elastic compression of the concrete. By (1) section 3, $\frac{1}{k}=\frac{nf_s}{f_c}+1$, and by (2), $k=\frac{2Af_s}{bdf_c}=2p\,\frac{f_s}{f_c}$, hence $\frac{1}{p}=\frac{2f_s}{f_c}\left(\frac{f_s}{nf_c}+1\right)$ from which values of p calculated from assumed values of f_s/f_c between 15 and 35 for $n=12$ and $n=15$ have been plotted in Fig. 49. Since the steel element in the combination is more dependable than the concrete from the standpoint of uniformity of strength, the safety of the structure is made by experienced builders to depend on the steel. In order to effect this the working strength of the concrete should be taken at a smaller fraction of its ultimate strength than the working strength of the steel is of its ultimate strength. For a 1:2:4 mix 650 pounds per square inch is a safe working stress to resist compression in concrete arising from bending. Both tension and compression are developed in concrete by flexure and by bond shear. The resistance, however which it offers to tensile stress is small compared with that which it offers to compression. Forty pounds per square inch is a safe value of the working tensile resistance.

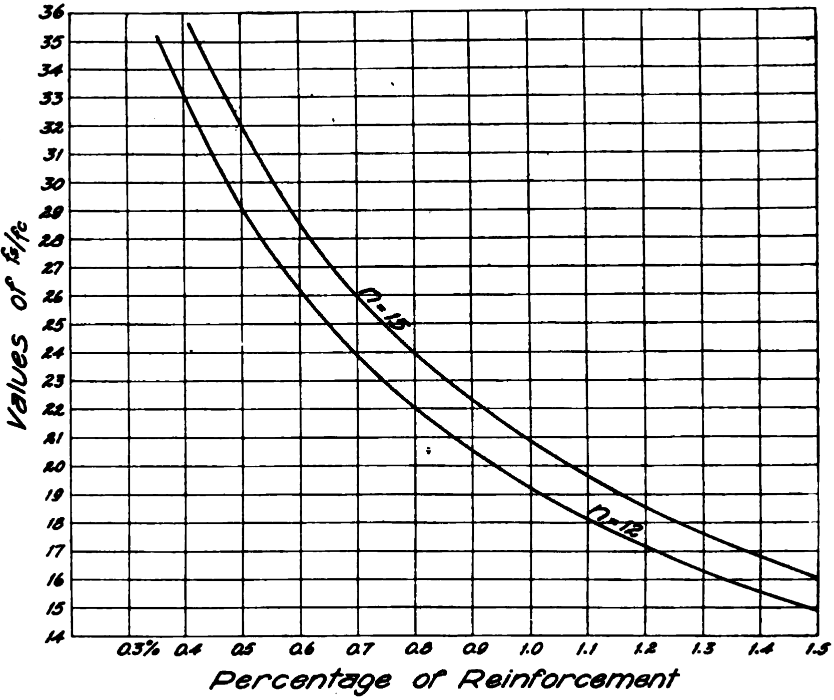


Fig 49. Limiting per cent of Steel for Different values of f_s/f_c.

Now if a beam is to depend for its stability upon the stress in the steel, that stress must not exceed a certain assigned value dependent upon the quality of the steel, or what amounts to the same thing a given quality of steel must not suffer a stress or a corresponding elongation at any point in excess of an assigned value. In order to compare beams of different depths and the same assumed maximum stress f_s or elongation e of steel, draw two plane sections of the beam under consideration at right angles to the neutral axis and at a distance of one unit apart. Before bending occurs, the two sections are parallel to each other, but after bending they make some elementary angle $\triangle \theta$ with each other. Draw thru one extremity of the unit of length at the neutral axis (which has been unchanged in length by bending) a plane parallel to the section at the other extremity of the unit length. It consequently makes an angle $\triangle \theta$ with the original plane section at this extremity. Any horizontal shearing deformation may be disregarded in this comparison because it will affect both sections to practically the same amount. The unit elongation of the steel due to the bending between these two unit sections will be $e=(1\text{-}k)d\triangle \theta$.

This investigation is made upon the assumption that e has a value which is constant and the same for different beams, but with the proviso that under this steel elongation neither the shearing distortion nor the compression of the concrete anywhere shall exceed permissible limits, questions which will have to be separately investigated since they depend on the steel stresses in too complex a manner to be readily introduced into consideration at the same time with the effect of the constancy of the steel stresses.

Now other things being equal $\triangle \theta$ decreases as p, the percentage of the reinforcement, increases; i. e. $\triangle \theta = c/p$ where c is an experimental constant whose value is dependent upon the grade of concrete, etc. Substitute this value of $\triangle \theta$ in the previous expression, then

$$e/c=(1-k)d/p = \frac{(1-k)L}{pN}$$

in case d be assumed to be some known fraction $1/N$ of the span L, *i. e.* or $d=L/N$.

It thus appears that the last member of this equation will be found to be an experimental constant for reinforced beams of the same span and grade of concrete; and in case the numerical value of this constant be determined for any given beam not liable to excessive deformation at the center, it will have the same value for a beam of different depth, span, and percentage of steel, provided, as before stated that sufficient resistance to compression and diagonal tension be supplied.

For example, assume $f_s = 13,000$ and $f_c = 650$, then $f_s/f_c = 20$. Referring to the curve for f_s/f_c Fig. 49 the corresponding steel ratio is $p = 0.0105$, and taking the corresponding value of k from the k curve, it appears that $(1-k) = 0.573$. It is known by experience that a beam whose depth is 1/12 of the length should have this amount of reinforcement, or $N = 12$ when $p = 0.0105$.

Hence $$\frac{(1-k)L}{pN} = \frac{0.573\ L}{0.0105 \times 12} = 4.55\ L$$
is the constant for such beams.

The same curve shows that for $f_s/f_c = 26$, $p = 0.007$ and $(1-k) = 0.635$; hence, using these and the constant 4.55, we find $N = 20$.

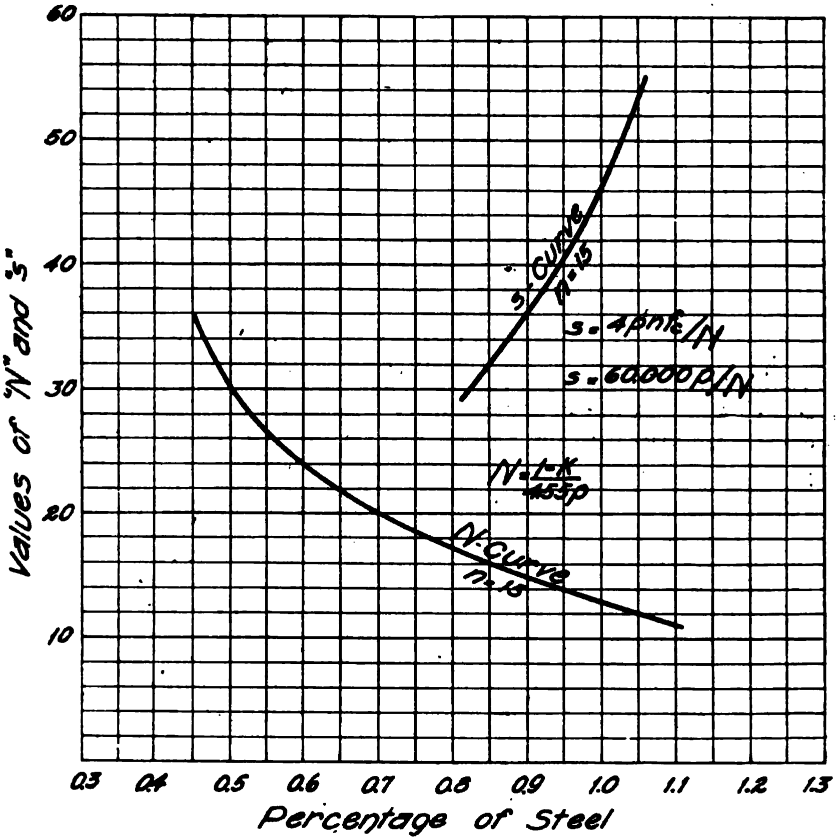


Fig. 50.

Again, for $f_s/f_c=32$, $p=0.005$ and $(1-k)=0.68$ and $N=30$.

The large values of f_s/f_c which have been assumed above for the shallow beams $N=20$ and $N=30$, reduce the working stress f_c and the steel ratio p below the values for $N=12$ in accordance with good practice. The accompanying diagram Fig. 29 gives values of $L/d=N$ for usual values of p computed from the equation $N=(1-k)/4.55\ p$ and $n=15$.

Now the total tension T in the steel at mid span of a simple beam is $T=WL/8jd$. But $T=bdpf_s$ in which f_s is the unit steel stress at mid span, hence $f_s=\dfrac{WL}{8\,jbd^2p}$.

But the identical elongations e of both the concrete and the steel at mid span may be written

$$e=f_s/E_s=f_c/E_c, \quad \text{or } f_s/f_c=n$$

and $$f_c'=\frac{WL}{8\,jbd^2pn}$$

in which f_c' is the apparent direct tensile stress in the extreme fiber of the concrete at mid span as shown by its elongation e while in contact with the reinforcement. The so called apparent stress f_c' may or may not correspond to an actual stress of some considerable amount. It is used here simply as another way of expressing the actual elongation e. The experiments of Considère* show that concrete when well reinforced may remain intact under elongations not only far in excess of any possible for concrete without reinforcement but in fact remain intact under elongations several times as great. The reasoning here employed is however entirely independent of any question of actual checking or not, for $f_c'=eE_c$ is simply a convenient unit of comparison computed as the product of elongation and modulus.

Next obtain the shearing stresses and the diagonal tension in the concrete at the extremity of a simple beam. The total horizontal shear between a unit of length of the reinforcement and the concrete is such that a segment of the beam lying between two vertical planes which are one unit apart is held in equilibrium by the total vertical shear $\frac{1}{2}W$ acting with the arm unity and the total horizontal shear S acting with the arm jd.

$$\text{Hence } \tfrac{1}{2}W=S\,jd, \text{ or } S=\tfrac{1}{2}W/jd.$$

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*Experimental Researches on Reinforced Concrete, McGraw Pub. 2nd Ed., p. 224.

This makes the unit horizontal shear on any horizontal plane below the neutral axis

$$s = S/b = \tfrac{1}{2}W/jbd = 4\,pd\,f_s/L = 4\,p\,f_s/N$$

provided the total shear in a unit length be regarded as uniformly distributed thruout the breadth b of the beam. This is equal to the unit diagonal tension at the end of the beam which is produced by the shear alone.

Hence $s = 4\,pn\,f_c/N$

Take $f_c = 1000$ and $n = 15$ then $s = 60{,}000\,p/N$

By using corresponding values of p and N given previously we find

p	N	s
.0085	16	32
.009	15	36
.0095	14	40.7
.01	13	46

Plotting the values of s corresponding to the assumed values of p it appears that s will reach a safe limiting value of 40 lbs. per sq. inch when $p = .0094$ nearly and when the span is somewhat more than fourteen times the thickness, or $L/d = N = 14.25$, as may also be seen from the diagram Fig. 50. Beams more shallow than this will have smaller values of s, but deeper beams where $N < 14.25$ will require reinforcing to resist diagonal tension at the ends, when there is a working stress of 16,000 lbs. per sq. inch on the steel at mid span.

6. Limiting Safe Shear for Simple Reinforced Concrete Beams: Where the safe vertical shearing resistance is not predetermined by the shallowness of the beam, percentages of reinforcement as high as 1½ percent may be used for beams of 1 : 1½ : 3 concrete, in which the ratio of depth to span is as low as 1/6 but for ratios of depth to span of 1/10 reinforcement should not exceed about 1.15 percent. Permissible values of the intensity of the vertical shearing stresses, $S = \tfrac{1}{2}W/jd$, at the end sections of concrete beams are shown in the diagram, Fig. 50. These values for shear may be increased somewhat where there are bent up bars and horizontal bars that have ample lap over the supports. The total cross section of the steel may then be figured at 6,000 pounds per square inch shear for bent up bars and half that for horizontal bars in addition to the shearing resistance of the concrete in developing an ultimate unit shear on the total cross section, which shear should never exceed five percent of the compressive strength of the concrete cured 90 days at 70 degrees F.

9. Systems of Simple Beam Reinforcement. The two methods of developing shear resistance commonly employed are by bending up a portion of the main reinforcing elements, and by the use of stirrups or vertical bars.

Fig. 51 represents the idea of some of the old time builders who feared, as late as the year 1905, to put continuous beams in a concrete building.

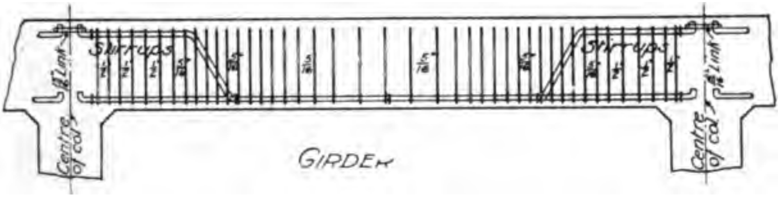
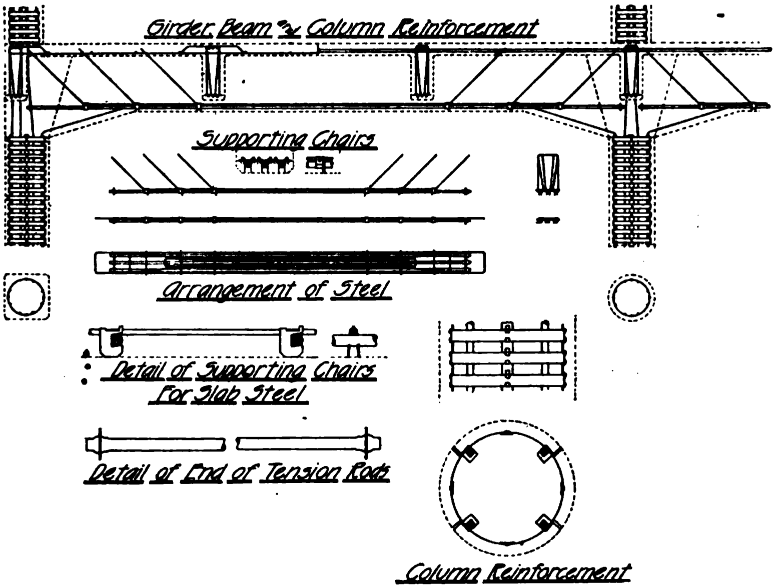


Fig. 51

The Cummings Beam is illustrated in Fig. 52 in which the bent up rods are double and form a loop extending into the top of the beam, which provides an efficient web reinforcement. The ends of the horizontal bars are upset and provided with a washer for end anchorage.


Girder Beam & Column Reinforcement
Supporting Chairs
Arrangement of Steel
Detail of Supporting Chairs For Slab Steel
Detail of End of Tension Rods
Column Reinforcement

. 52

In the Kahn reinforcing bars, the so-called shear members are bent up at an angle of 45 degrees to the bar, The body of the bar is rolled with flat fins projecting from the diagonally opposite corners of the square body. These fins are later sheared longitudinally and bent up as shown in Fig. 53.

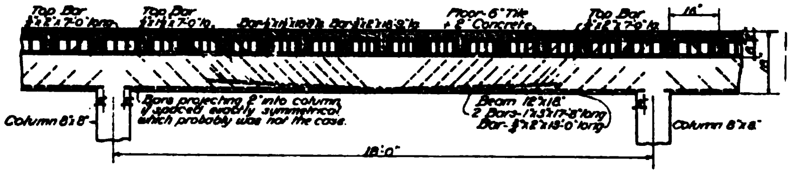

Top Bar
Top Bar
Floor-6" Tile + 2" Concrete
Top Bar
16"
Column 8"x8"
Bars projecting 2" into column if spaced exactly symmetrical which probably was not the case.
Beam 12"x18"
2 Bars-1"x3"x17'-8" long
Column 8"x8"
18'-0"

Fig. 53

The particular design in the sketch Fig. 52 which shows the bent up shear members exhibits serious defects, equally bad in this or any other kind of reinforcement, to wit, insufficient lap of the steel over the supports.

In the tests of Prof. Brown, Proceedings Canadian Society of Civil Engineers, October 15, 1908, he found more satisfactory results with bars bent in the usual manner than with the particular Kahn bars used in his test beams.

Fig. 48 showing Withey's tests of simple beams shows a common method of bending up main reinforcing bars.

C. A. P. Turner prefers to follow the Bollman type, Fig. 54, in the distribution of bent up steel with ⅜" round stirrups spaced about 12" apart at the end and 20" at the center.

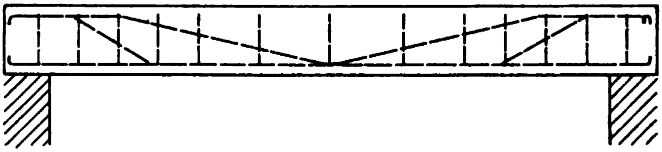


Fig. 54

The advantage of this type over those before illustrated is that by this arrangement diagonal tension through the web is well provided for at all points in the length of the beam by the main bent up rods.

Morsch, Reinforced Concrete, page 153, shows three test *T*-beams built with horizontal bars in the bottom throughout, one half of the beam being reinforced with stirrups, the other half unreinforced. He compares these with a beam V, p. 163, in which two of the four main reinforcing bars were bent in catenary style from the third point of the beam, stirrups being used in half of the beam and omitted from the other half. From a comparison of these test specimens he concludes that "the suspended system of reinforcement does not give any increase of safety against the appearance of diagonal tension cracks or the final failure produced by them as compared with straight rods without stirrups and that stirrups are so much the more necessary."

The tests of similar *T*-beams having four rods; with three rods bent up more sharply at three points toward the end, produced greater strength than with two rods bent up, and between the two beams with rods bent up at these points that in which the single bottom rod was hooked at the end gave an ultimate strength ten percent higher than that in which the rod was run through the beam without hooking.

The fundamental theory of reinforced concrete is to provide steel to resist the tensile strains and tensile flange stresses and to utilize the concrete in resisting compression. All of the foregoing arrangements of reinforcing metal are to a considerable extent deficient in the complete accomplishment of this desired end, with respect to the tensile web stresses.

M. Cottançin, the French constructor, has developed a type of beam reinforcement in which the tensile web stresses developed by shear and direct tension are admirably provided for. His arrangement consists essentially of a net work of closely spaced ¼" rods woven in and out, these rods running lengthwise of the beam and vertically. This vertical mat of reinforcement to resist diagonal tensile deformation and strengthen the concrete under vertical shear has rendered possible the construction of roofs of astonishing lightness and beauty.

The saving in the amount of material, however, is at the expense of such a large amount of labor as to render this method of construction commercially unfeasible in the United States under present conditions. It is worthy, however, of the most careful study by the expert in this line.

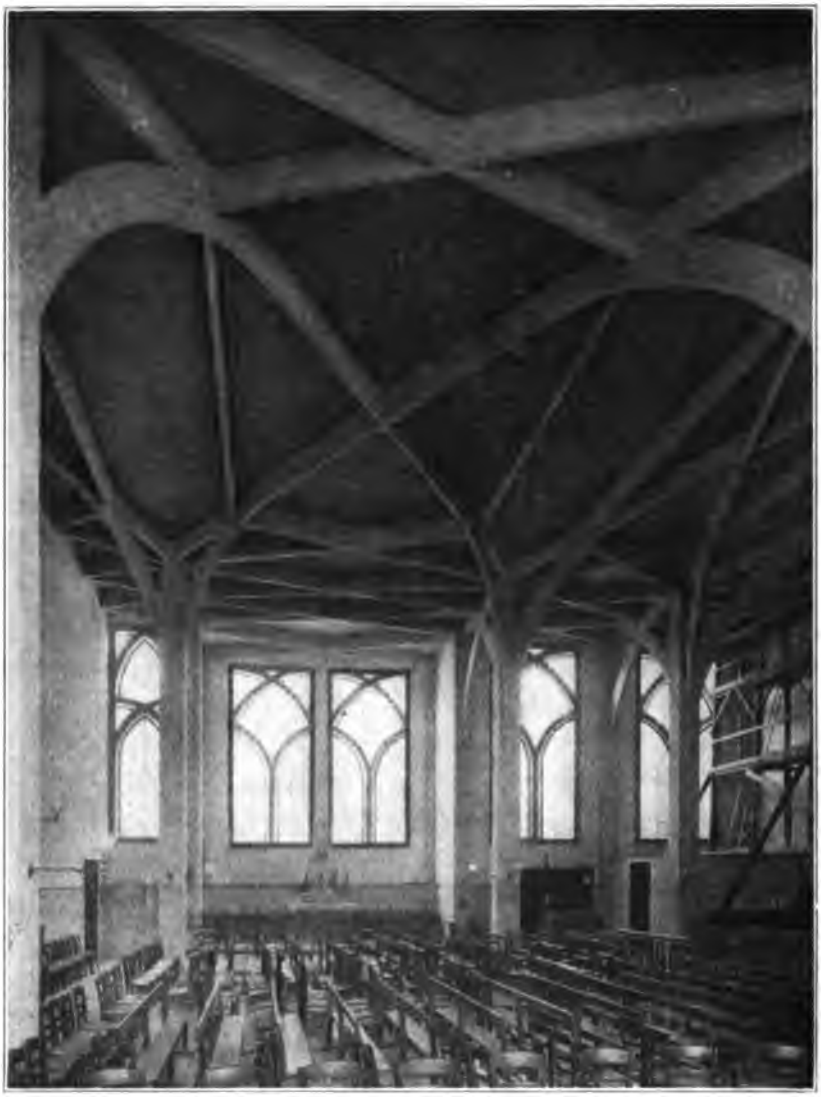


Fig. 55. Cottancin Beam Construction in the Church of St. Jean de Montmartre

Fig. 55 shows his beam construction in the Church of St. Jean De Montmartre, and Fig. 56 is a view of the work under construction. The reinforcement which he used is illustrated in Fig. 57 from the work of Marsh.

Of Cottançin construction, Marsh says:

> "The remarkable strength of Cottancin's ribs is demonstrated by the fact that whereas a ribbed reinforcement with 26.4 pounds of metal will bear a load of 200 pounds per foot with only a powdering of the concrete, the same amount of metal in the form of a rolled joist will fail under a load of 33 pounds per lineal foot, the span in both cases being 14.76 feet."

Marsh considers that the principle of Cottançin is based upon prevention of molecular deformation by supplying resistance of the reverse kind for stresses on small particles which produce noticeable results and very light structures.

The carrying capacity of the Cottançin beam as Marsh has pointed out gives results not accounted for on the basis of the usual theory but which may be accounted for readily when we consider the scientific manner of reinforcing the web so that the indirect stresses are provided for in a manner which does not over-tax the concrete and so the loss of energy accompanying the inelastic deformation of the ordinary beam is avoided.

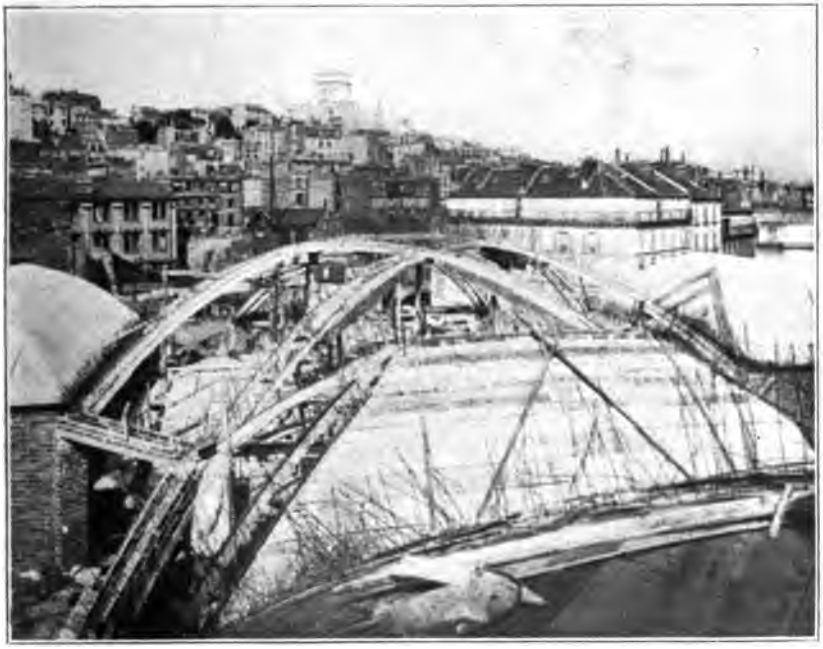


Fig.56. Cottançin Beam Construction in the Church of St. Jean de Montmarte.

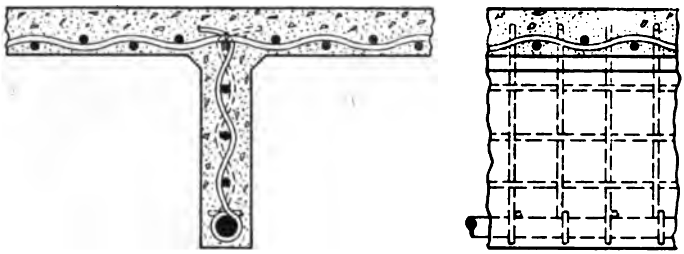


Fig. 57. Cottancin Beam

10. Discussion of Elastic Properties of Beams and the Assumptions involved in the Preceding Theory. In discussing the elastic properties of concrete it was shown that the modulus of elasticity of the concrete is not the same for different loads and further that the modulus changes with the age of the concrete, it being only $\frac{5}{8}$ as great at the age of thirty to forty days as it is at the age of two or more years, and furthermore that while this modulus is usually considered the same for tension and compression, this is open to some question. Accordingly a reasonable approximation to the conditions which occur in bending when the building is first ready for occupancy is assumed and in nearly all building codes the ratio of the modulus of elasticity of steel to concrete in bending is assumed at 1 to 15. A difference in this ratio would affect the position of the neutral surface to some extent and the effective lever arm to a still smaller extent, as shown by comparison of the j and k curves in the diagram in Fig. 46. The assumption of the value of n as 15 for bending is accordingly on the safe side, and the error involved is not great. On the other hand, this divergence of practical conditions from the assumptions used in the computation do not justify a high degree of mathematical precision in the work of practical design, for if the computations are carried to a degree of nicety unwarranted by the accuracy or agreement of the assumption with practical conditions it is a mere expenditure of time without commensurate results. Accordingly, it may be stated that the approximate formulas for beams and slabs are sufficiently accurate for practical purposes.

A convenient diagram for computation when $n=15$ is presented in Fig. 58, and a similar diagram when $n=12$, in Fig. 59.

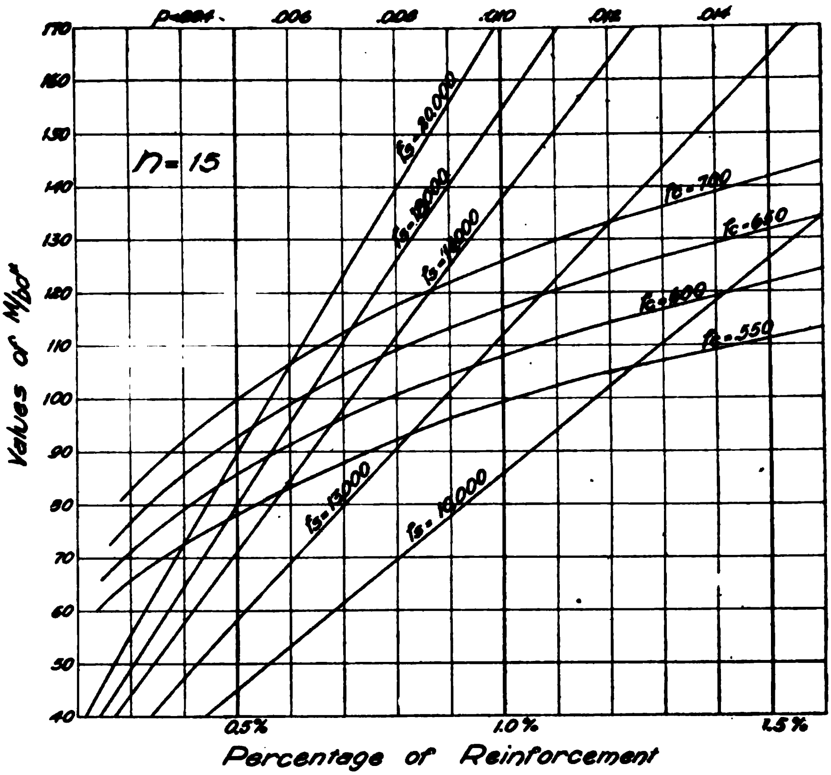


Fig. 58. Diagram of Steel Percentages for Different values of $M \div bd^2$

In *T*-beams where the beam is integral with the slab, it is customary to assume a width of slab not exceeding four times the slab thickness as forming a part of the compressive flange of the beams. This assumption, of course, is conservative. It is evident that the compression in the outside edge of that portion of the slab which is regarded as useful section is less than portions nearer the axis of the beam, and that the slab beyond this imaginary division is also restrained in compression, the condition approaching what has been designated as the "twilight zone" between exact knowledge and conjecture as to the actual conditions. Evidently in a case like this, exact computations beyond the limits of accuracy of the assumption is a waste of time and the effective depth jd of a *T*-beam may be for practical purposes determined at once by the assumption of $(1\text{-}j)d = z = \frac{3}{8}\ t$ Fig. p. 157 without material error when d lies between 2.5 t and 4 t.

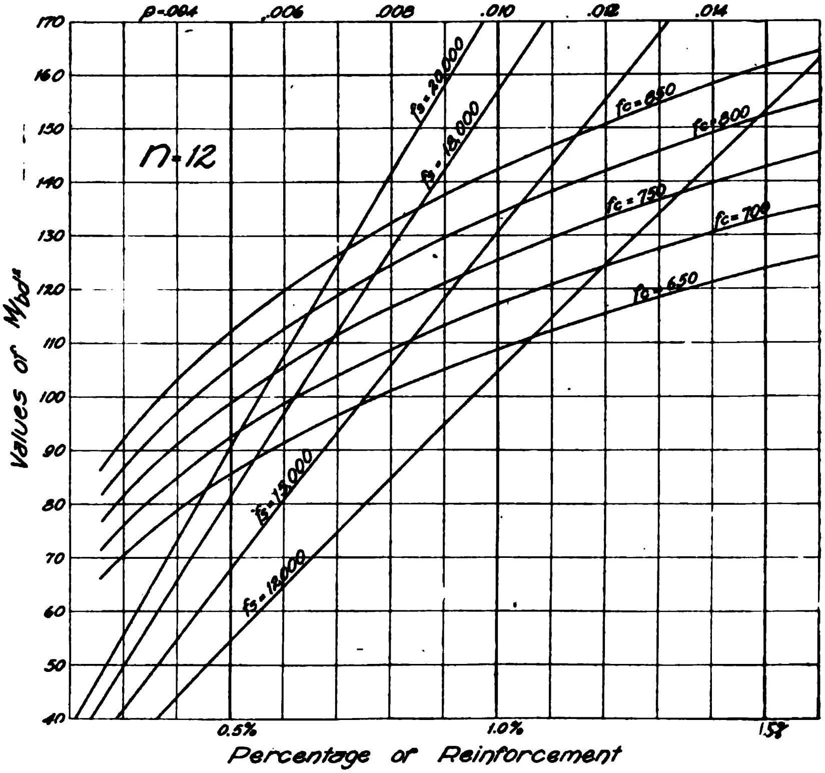


Fig. 59

11. Moment of Inertia. The moment of resistance M of a reinforced beam calculated from the total tensile resistance of the steel Af_s acting with the arm jd about the centroid of compression is $M = Af_s\, jd$. Now in the theory of flexure of beams one factor of M is the unit stress at a distance of one unit from the neutral axis, viz., f_s/id.

When so written we have $M = Aijd^2 . f_s/id$, or $M = If_s/id$, in which the factor $I = Aijd^2$ is defined as the "moment of inertia" of the cross section of the beam, consisting entirely of a summation of geometrical elements as follows; the product of each element of the area A on which the tension occurs multiplied by its distance id from the neutral axis and that multiplied by its arm jd from the center of compression. The reason why the compression area is not also included in this summation is this; the sum of the products for the compression areas vanishes because their arms are measured from their center of gravity. Since the resistances form a couple their moment may be taken about any axis whatever without altering the moment.

CHAPTER VII

MECHANICS OF RESTRAINED AND CONTINUOUS BEAMS

1. **The classification of Beams** is based upon the manner in which the beam is supported thus:

A simple beam is one which is merely supported at the ends, and its mathematical treatment is based upon the consideration that the beam is free at the end to turn and that the supports offer no resistance to rotation. Such a beam, does not, of course, exist in practice, but all beams which rest upon supports and are not rigidly restrained or have only a small degree of restraint at supports are treated from the practical standpoint as simple beams and figured as such.

Concrete beams or slabs which are apparently continuous over supports but which have reinforcing metal at the bottom thruout, offer such small resistance to negative moment at supports that they are treated on the theory of predominant action as simple beams unless their depth be so large relatively to the span and the horizontal thrust afforded by the supports be such that they may be treated somewhat on the arch principle.

Where, however, a beam is continuous thruout, and rigidly built into and integral with a series of columns, and suitable reinforcement is provided at the top of the beam over the supports which extends outward to the line of inflection, and then thruout the bottom of the beam, we have a true continuous beam in which the bending moments over supports follow the laws of continuous beams except as they are modified by the rigidity of the integral union or connection with the columns.

The effect of this monolithic connection is to cause the deportment of the beam to approach more and more nearly, for all spans, to the condition of a continuous beam extending through an indefinite number of spans; in other words, to cause the moment over the support for uniform load to become $WL/12$ and the moment at mid span $WL/24$. An end span of the series, however, except in heavy warehouse construction will not receive this full degree of restraint. In a heavy warehouse with the large columns, 26 inches and over in diameter, this degree of restraint is, for practical purposes fully secured, but with smaller columns it may be less, and its amount is to be determined approximately by the designer from comparison of the relative rigidity of the columns and beams, so that in the case of light columns the spans that end at the wall should be somewhat more heavily reinforced for moment at mid span.

The case of an unloaded span with both adjacent spans fully loaded is not uncommon in a warehouse and this, too, must be provided for.

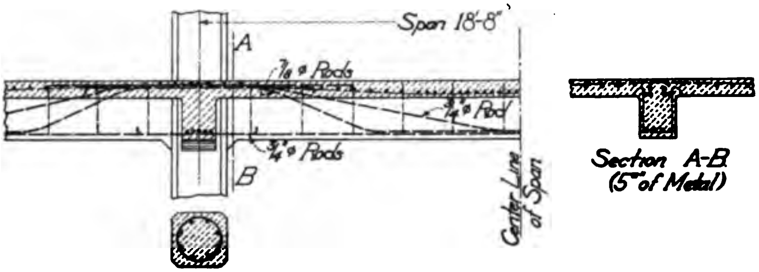


Fig. 60. Continuous Beam, Turner System.

Where the construction consists of beams in but one direction with the slab spanning from beam to beam and with insufficient metal parallel to the beam in the slab to fully reinforce the beam to resist the negative moment under the circumstances stated, which may result from the excess of live load stress over and above the dead load stress, the beam should be treated, in determining the central moment, as continuous for dead load only and as a simple beam for the live load.

The preferable arrangement, however, is the provision of beams in both directions from column to column where beam and slab constructions are used, making the floor a true monolith or a natural concrete type. For this type of construction with ordinary spans, beam reinforcement consisting of say five rods arranged as indicated in Fig. 60 is preferable, in which the beam rods consist of two which extend thruout the length of the beam at the bottom and into the adjacent span, two which are bent up from the quarter point to the top of the beam and extend over into the adjacent span, and one which while extending into adjacent spans slopes gradually from the top of the support to the bottom of the beam near the center of the span.

An arrangement of this kind, after the manner of the Bollman truss, furnishes liberal provision for shear at the support, while the inclined rods, under bending strain resist shear thru their inclination at the support. It is only necessary to figure the moment over the support as $WL/12$ and provide therefor by the cross section of the six rods crossing it while the four rods in the bottom of the beam and the five rods at the center are ample for all possible conditions of loading. The lap of rods at the support and beyond the support both at the top and bottom render sudden failure or collapse practically impossible after the concrete has had even a few hours under normal temperature conditions in which to harden.

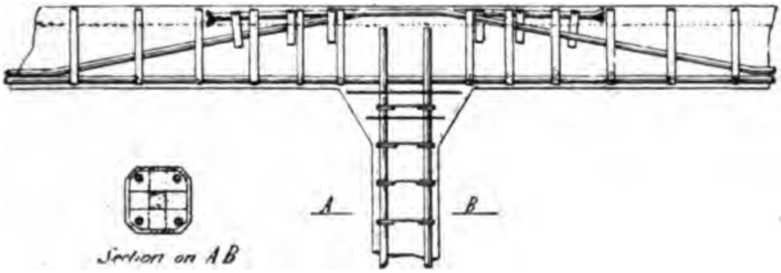


Fig. 61. Continuous Beam, Hennebique System.

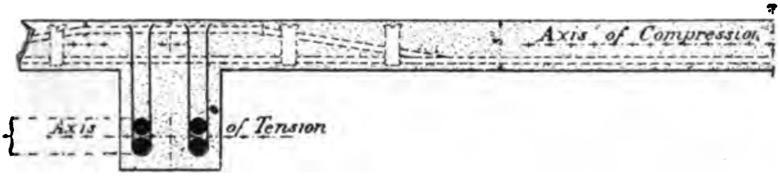


Fig. 62. Cross Section of Beam and Slab.

In addition to enhanced safety there is very material economy in such an arrangement, since compared with beams of constant section, the continuous beam is more than five times as stiff and one and one half times as strong as the simple beam having the same cross section of metal thruout the bottom of the beam as the continuous beam has at the top over the support.

Fig. 61 shows the Hennebique continuous beam which has an enviable record from the standpoint of safety by virtue of the liberal lap of reinforcement and stirrup verticals employed.

The problem that confronts the designer of a monolithic continuous beam is one of correctly proportioning it to resist the applied moments over the supports and between the supports in view of the relative rigidity of the supports which may be insufficient to completely restrain the beam so that it will remain horizontal over the supports under unbalanced loading.

If the supports are yielding so that this tangent varies from a horizontal position then there is a change in the distribution of moments along the beam and a change in the reactions at the supports, all of which is determined by the principle of rigidities applied to the relative stiffness of the beam and the supporting member or structure. This will be treated at length in Chapter XII.

2. Differential and Summation Equations in Beam Theory.

(a) Let w be the distributed loading in lbs. per unit of the span L of a beam, and if S expresses the total shearing stress acting at any vertical right section, then if x be its abscissa.

$$\frac{dS}{dx}=w \quad \text{-----------------------------------} (1)$$

If the load be uniformly distributed w is constant and the load curve will be horizontal thruout as shown in Fig. 63.

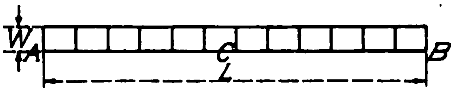


Fig. 63

The shear curve which has a uniform slope may then be expressed by the equation:

$$S=\Sigma(w) \quad \text{-----------------------------------} (1)$$

in which dx is taken as unity.

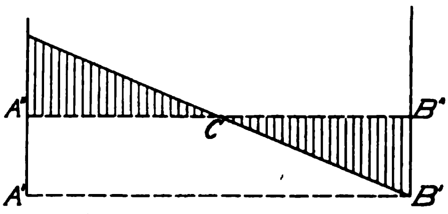


Fig. 64

If the span be subdivided into equal segments dx and the summation of the ordinates w begin at B' in Fig. 64, and the ordinates S are laid off from $A'B'$, then the values of S are those of a cantilever fixed at A'. The same slope is obtained if the summation begin at C, when S is measured from $A''B''$, but the ordinates S between it and the sloping shear curve then represent those in a beam equally supported at A'' and B'', either freely or fixed, *i. e.*, the position of the line of zero shears with respect to the shear curve depends upon the manner of support, but the curve itself which depends on the loading only, remains unchanged, while dx is taken as $+1$ at the right of C and as -1 at the left of C.

(b) Again, if M is the bending moment at any right section of the beam

$$\frac{dM}{dx}=S \quad \text{(2)}$$

and the moment curve may be expressed by the equation

$$M=\Sigma(S) \quad \text{(2)}'$$

and if we construct the moment curve in Fig. 65 by summing the ordinates of the shear curve Fig. 64 and laying them off from a zero line we shall arrive at the same moment curve whether we use the zero line AB or $A'B'$ Fig. 65. Take $A'B'$ as the zero line and begin the summation of S at B'' Fig. 64, then at C Fig. 65 we have $\Sigma S=C'C$. At the left of C' the shear S changes sign and the ordinates M consequently decrease to zero at A' Fig. 65. Again, suppose the summation to begin at C Fig. 64, then the ordinate M is zero at C Fig. 65 and the line of zero ordinates Fig. 65 is ACB, corresponding to a beam with ends built in and fixed at A and B and cut in two at C but with precisely the same moment curve as before. Were the line of zero moments A_0B_0 so drawn as to have equal positive and negative areas between it and the moment curve in each half of the beam, that would correspond to a single span of a continuous beam restrained horizontally at the supports in which $2CC_0=C_0C'$.

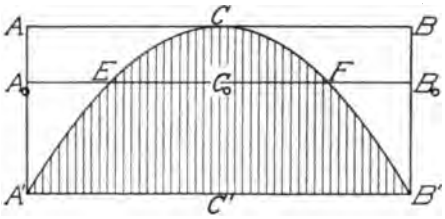


Fig. 65

Let b be the amount of bending produced by the moment M per unit of length of the beam when measured in radians; then since we are dealing with material that obeys the laws of elasticity,

$$b=(B)M \tag*{(b)}$$

in which the bending coefficient depends only upon the properties of the material and the dimensions of the cross section. In fact, $(B)=1/EI$, and the ordinates of the moment curve consequently represent, on a vertical scale different from that used for M, the sharpness of the curvature of the beam, which is also expressed by $1/R$ provided R is the radius of curvature of the beam. Hence the curvature

$$b=1/R=M/EI \tag*{$(b)'$}$$

is represented by the ordinates of the moment curve Fig. 65 when measured on a suitable scale.

(c) Again, by geometry the angle of slope a of the deflection curve is related to the bending b by the differential equation

$$\frac{d\,a}{d\,x}=b \tag*{(3)}$$

or by the summation equation

$$a=\Sigma(b)=\Sigma(M)/EI \tag*{(3)$'$}$$

But if we adopt a suitable scale for the moment curve we may write

$$a=\Sigma(M) \tag*{(3)$''$}$$

Whenever we are concerned with the relative and not with the absolute values of either one of these quantities such as w, or S, the scale is of no consequence. For example, we assumed w to be constant, but did not assume what its absolute value might be, and consequently have found only relative values of S, M and b.

If now the summation in (3)$''$ be begun at C' of Fig. 65 and if we lay off $\Sigma(M)$ from $A''B''$, the slope curve will have the shape shown in Fig. 66, crossing $A''B''$ at C'' and horizontal at the ends, and the slopes will be those of a beam simply supported at the ends.

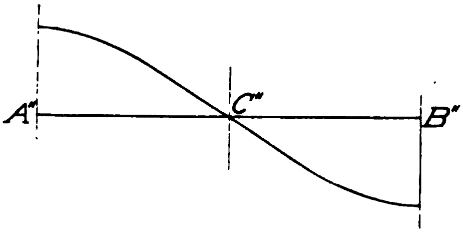


Fig. 66

Again, if the summation in (3)″ be begun at C_0 of Fig. 65 as the middle of a span of a continuous beam fixed horizontally at the supports, the slope curve would have the shape shown in Fig. 67, where the slope vanishes both at mid span C_0 and at the ends A_0 and B_0.

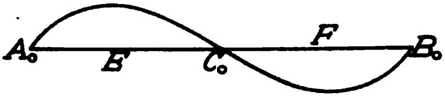


Fig. 67

Now, if the summation in Eq. (3)″ be begun at A or B, Fig. 65, and the ordinates M be measured from AB, the values of a so obtained will express the slope of two separate cantilever beams AC and BC with free ends at C and fixed ends at A and B. The curve whose ordinates represent the slope of these beams has the following shape shown in Fig. 68.

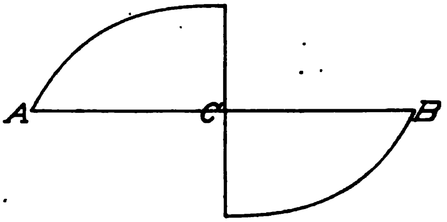


Fig. 68

Other cases would be treated in a similar manner.

Now an initially plane right section of a beam will not after flexure has occured, be in general, either vertical or normal to the curve of flexure but will lie between the normal and the vertical. The angle between the normal and the vertical is equal to the slope a, and the angle between the normal and the plane section is the horizontal shearing distortion h or detrusion at that section, and is measured in radians. By reason of the elastic properties of the beam h bears the same ratio to a at any and all sections along a uniform beam, or

$$h=(A)\,a \quad \text{..................................} \quad (a)$$

in which (A) is a constant for a given beam. Consequently the curve whose ordinates are equal to the horizontal shear h, is the same as the slope curve for a, when a suitable scale is employed.

(d) Finally, in case y is the ordinate of the deflection curve of a beam of which the angle of slope is a, we have

$$\frac{dy}{dx}=a \qquad (4)$$

and

$$y=\Sigma(a) \qquad (4)'$$

Let the summation in $(4)'$ begin at C'' Fig. 66, then the shape of the deflection curve of the simple beam supported at its ends, is that shown at the bottom of Fig. 69. It is very nearly parabolic in shape, but has a smaller radius of curvature at mid span and straighter curved sides than a parabola of the same vertical height and span.

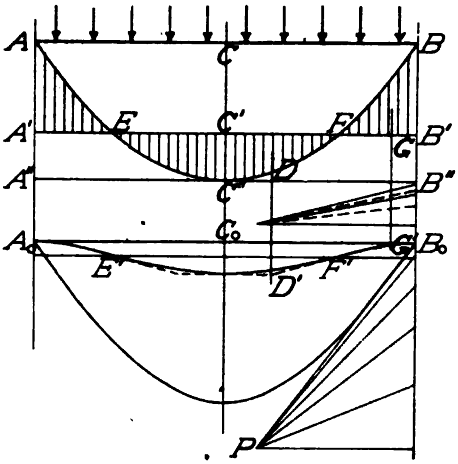


Fig. 69

But if the summation in $(4)'$ begins at C_0 of Fig. 67, and A_0B_0 is the zero line of the ordinates the summation gives the deflection curve for y passing thru the points E' and F' for a span fixed horizontally at the ends as shown in Fig. 69.

The points E and F are the two points where the moments vanish, and the slope of the beam is greatest at the points of inflection E' and F' in the deflection curve.

(e) The method of a simple summation by which the foregoing curves have been successively derived and laid off each from the other may be regarded as a graphical process by which they may be approximately constructed. But there is another and a different graphical process applicable to several of these curves by which one may be derived from another, for by (1) and (2) we have

$$\frac{d^2M}{dx^2}=\frac{dS}{dx}=w \quad \text{..............................} (5)$$

which expresses the relation existing between M and any load w. Now it is known that the curve of moments is the curve of equilibrium, or the shape assumed by a flexible wire which supports the load w, as shown in Fig. 65. Such an equilibrium curve can be readily constructed graphically by a systematized application of the parallelogram of forces. An equilibrium curve so constructed will consequently be the moment curve of the load w, and the same method may be applied to obtain and represent the relationship of any two quantities one of which is the second differential coefficient of the other.

Hence if the moment curve be constructed which is regarded as due to some kind of distributed loading whose intensity is M, the equilibrium curve of this loading having ordinates by equations (4) and (3)′ equal to

$$\frac{d^2y}{d\,x^2}=\frac{d\,a}{dx}=M/EI \quad \text{..............................} (6)$$

will, when measured on a suitable scale, be a moment curve in which the actual moments are taken as loads, whose ordinates y will be the ordinates of the deflection curve, (Mohr's theorem).

This relationship of deflection curves of a given beam when subjected to different restraints at supports is well illustrated in the constructions in Fig. 69 where the exaggerated deflection curves are shown, of a beam simply supported, and of the same beam built in horizontally at its ends, in which it is evident that both are drawn to the same vertical scale because the scales of both load areas and both pole distances are the same. The constructed central deflection of the restrained beam is seen to be one fifth that of the simple beam as it is otherwise known that it should be. The fourth degree deflection curve of the simple beam differs relatively little from the parabola with which it may be compared.

Again, let S Fig. 64 be regarded as a kind of distributed loading. It is related to h by the equation

$$\frac{1}{(A)}\frac{d^2h}{dx^2}=\frac{d^2a}{d\,x^2}=\frac{db}{dx}=\frac{1}{EI}\frac{dM}{dx}=\frac{S}{EI} \quad \text{..............} (7)$$

It therefore appears that when suitable scales are employed *i. e.* any arbitrary scales whatever, the ordinates of a moment curve of the vertical shears regarded as a kind of loading represents the distribution of the horizontal shearing distortions, or the detrusion, and the slopes. This was represented in Fig. 66 for a simple beam, and in Fig. 67 for a beam fixed horizontally at the ends.

By Hooke's Law, the elastic horizontal shearing distortion measures the magnitude of the horizontal shearing stress, and there is no warrant for the ordinary statement that the vertical and horizontal shearing stresses are equal, because the supposed demonstration of that equality depends upon the equality of the opposite elementary couples produced by those stresses, thus implicitly assuming that no other couples act upon an element of the beam. Such an assumption is in general incorrect tho correct at mid span, because at other points the horizontal tensions and compressions which vary as their distance from the neutral axis produce a couple acting on each element in combination with which the shears must act to produce equilibrium.

How far wide of the truth the ordinary theory must be, that horizontal and vertical shears are equal may be seen from comparing the horizontal and vertical shears at and near the ends of a simple beam with those in a beam fixed horizontally at the ends. The vertical shears are the same in both as measured from $A''B''$ in Fig. 64, and consequently by the common theory the horizontal shears should also be the same *i. e.* the ordinates in Fig. 66 should according to this theory be the same as those in Fig. 67. Also see Section 7 Chapter III.

In the simple beam the horizontal shear and slope are greatest at the ends as shown in Fig. 66 but in the fixed beam as shown in Fig. 67 they are zero at the ends and greatest at the points of inflection.

(f) The deflection of uniform beams may be considered graphically from still another point of view of great simplicity and convenience. By equation $(b)'$ it appears that bending or change of slope per unit of span is proportional to the bending moment area M applied to that unit of span. But a given amount of moment area will produce theoretically the same total amount of bending (*i. e.* variation of slope) whether it be distributed along one unit of span or along many units or concentrated at a point, and a moment area will produce the same deflection at any point beyond the end of the distributed moment whether it be taken to produce bending as actually distributed along the span in the moment curve, or be taken to produce all its bending at the center of gravity of the moment area.

For example, consider the case of a uniform cantilever AB fixed at B and supporting a terminal load P at A. The bending moment increases uniformly from A to B and is represented by the ordinates between AB and AB'. If these ordinates be taken to represent the depth of some kind of loading, the moment curve of this loading will be some curve $A''B''$ which will be the exaggerated deflection curve of the cantilever in which the applied bending moment is distributed along the beam as represented by AB'. Now assume the entire amount of the bending moment to be concentrated and applied instead at G the center of gravity of the moment area so that all the bending would occur at the point G'. Were this possible the deflection and slope at A'' would be the same as that found when the moment was distributed along AB as first shown. But the slope and deflection would not be the same at any other point except B'' because the load applied at G is that of the entire moment area, and it must all be used in both the cases compared.

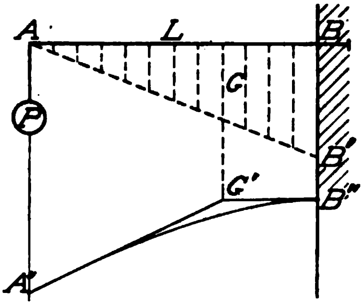


Fig. 70

Any arbitrary rearrangement, whatever, of the applied moments will give some configuration of false deflection polygon as $A''G'B''$ may be called, which will give the same slopes and deflections as the exaggerated deflection curve does at those particular points A'' and B'' where the slopes and deflections are the effects of the same total moment areas.

For example, in Fig. 69 of the curve of deflections of a beam fixed horizontally at both ends, the negative moment area BFB' shown in Fig. 69, near one end when concentrated and applied at its center of gravity G will produce a sudden change of slope at that point which will make the false deflection polygon coincide in direction and position with the exaggerated deflection curve at F' the point of inflection of the beam. Again, the equal positive moment area $FC'C''$ shown in Fig. 69 as distributed along the beam between the point of inflection and the mid span when applied at its center of gravity D will produce an equal sudden change of slope at that point in the reverse direction, and so make the side of the false polygon at mid span to be horizontal and tangent to the deflection curve, because the slope and deflection at mid span must be the same at that point whether we consider all the moment areas on one side of the center as concentrated or distributed in any manner. It will be more convenient, however, to consider the operation of a negative moment area $ACC'A' = \frac{1}{2}ML$ in which $M = AA'$ is the negative moment at A and $L = AB$, combined with the positive moment area $ACC'' = \frac{1}{2}L \times \frac{2}{3}a$ in which $a = C''C$. In order that the slope may vanish at the center the algebraic sum of these moment areas must vanish, hence $M = -\frac{2}{3}a$, which leaves $C'C'' = \frac{1}{3}a$.

3. *Moments Over Successive Equidistant Supports of a Continuous Beam Uniformly Loaded:* Let M_1, M_2 and M_3, or M_{n-1}, M_n and M_{n+1}, be the negative bending moments over three successive supports, where we may desigante the supporting reactions as R_1, R_2 and R_3, or R_{n-1}, R_n and R_{n+1}. Let the moment areas due to these moments and to the moments of the uniform loading be as represented in Fig. 71 in which T_1T_3 is tangent to the beam at the middle support R_2. Then the amount of deflection at R_1 from the slope line T_1T_3, which is produced by all the bending moments at the left of R_2 is some quantity $d = \alpha L$ while that produced at R_3 by all the moments at the right of R_3 is $-d = -\alpha L$.

According to the principle already developed these deflections may be taken to be produced as follows:

The moment area $AA'B = \frac{1}{2}M_1L$ produces an amount of deflection from T_1B due to the distance of its center of gravity from A, viz. $\frac{1}{3}L$, hence a deflection measured by their product $\frac{1}{6}M_1L^2$. Similarly the deflection $\frac{1}{2}M_2L$ due to $A'BB'$ is $\frac{1}{3}M_2L^2$, and that due to the parabolic area $AC'B$ is $\frac{1}{3}aL^2$ when $a = WL/8$.

Hence we have the equation of deflection

$$\frac{M_1L}{2} \times \frac{L}{3} + \frac{M_2L}{2} \times \frac{2L}{3} + \frac{2aL}{3} \times \frac{L}{2} = \alpha L = d$$

$$\frac{M_3L}{2} \times \frac{L}{3} + \frac{M_2L}{2} \times \frac{2L}{3} + \frac{2aL}{3} \times \frac{L}{2} - \alpha L = -d$$

Add these equations and cancel out L^2, then

$$\frac{1}{6}(M_1 + 4M_2 + M_3) + \frac{4a}{6} = 0$$

Hence

$$\frac{M_1 + M_3}{4} + M_2 = -a$$

$$\frac{M_2 + M_4}{4} + M_3 = -a$$

$$\frac{M_3 + M_5}{4} + M_4 = -a$$

- - - - - - - - - - - -

$$\frac{M_{n-1} + M_{n+1}}{4} + M_n = -a$$

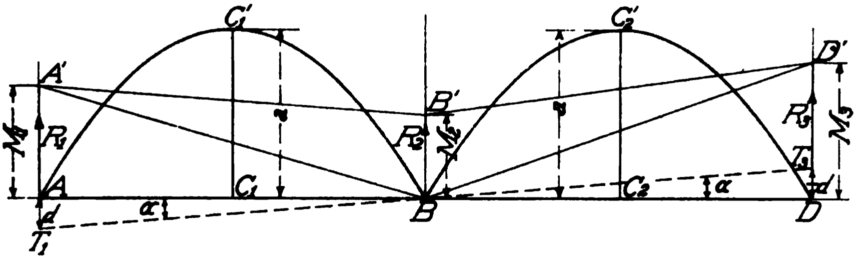


Fig. 71

This is the equation of three moments for equal spans and uniform loads which shows that for a beam of two equal spans and ends not restrained, we have, since $M_1 = M_3 = 0$, $M_2 = -a = -WL/8$; and for a beam of three equal spans since $M_1 = M_4 = 0$ and by symmetry $M_2 = M_3$ we have $\frac{1}{4}M_2 + M_3 = \frac{5}{4}M_2 = -a$.

Hence $M_2 = M_3 = -WL/10$.

Every pair of successive spans in a continuous beam gives rise to a single equation connecting the moments over the center and two end supports of this pair. These equations are sufficient in number to completely determine the values of all the moments at the intermediate supports of any continuous beam resting on level supports, *i. e.* when the end moments vanish.

4. Shears at Equidistant Supports in Continuous Beams: The load in any span is equal to the numerical sum of the end shears, *i. e.*, $S_1 - S'_2 = wL$ in which S'_2 is intrinsically negative where a prime is used to mark a shear at the left of a support. In case of a single span

$$S_1 = -S'_2 = \tfrac{1}{2}\, wL$$

and in case of two equal spans

$$S_1L = \tfrac{1}{2}wL^2 + M^2 = \tfrac{1}{2}wL^2 - \tfrac{1}{8}wL^2 = \tfrac{3}{8}wL^2$$

and

$$S'_2 = S_1 - wL = -\tfrac{5}{8}wL$$

The shears at the supports of continuous beams of multiple equal spans are each some fractional part of wL. These fractional multipliers or shear coefficients are what the shears reduce to when the span L and the load w per unit are each taken as unity. They are tabulated in the accompanying diagram, those at the right and left ends of any span being placed at the right and left sides of the vertical shown at the middle of that square, those at the right being negative.

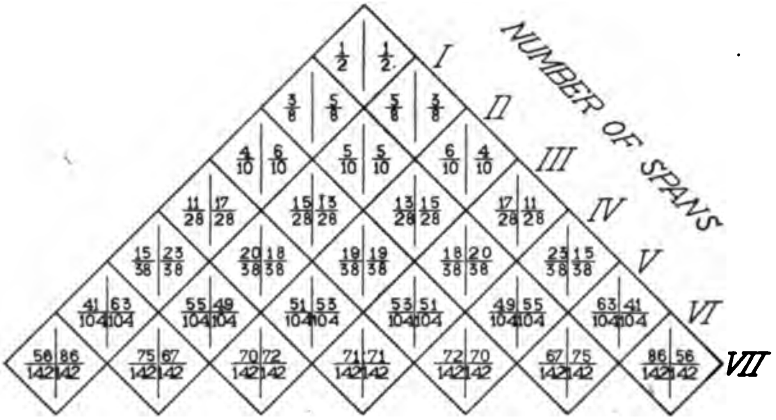

NUMBER OF SPANS
I: 1/2 | 1/2
II: 3/8 | 5/8 — 5/8 | 3/8
III: 4/10 | 6/10 — 5/10 | 5/10 — 6/10 | 4/10
IV: 11/28 | 17/28 — 15/28 | 13/28 — 13/28 | 15/28 — 17/28 | 11/28
V: 15/38 | 23/38 — 20/38 | 18/38 — 19/38 | 19/38 — 18/38 | 20/38 — 23/38 | 15/38
VI: 41/104 | 63/104 — 55/104 | 49/104 — 51/104 | 53/104 — 53/104 | 51/104 — 49/104 | 55/104 — 63/104 | 41/104
VII: 56/142 | 86/142 — 75/142 | 67/142 — 70/142 | 72/142 — 71/142 | 71/142 — 72/142 | 70/142 — 67/142 | 75/142 — 86/142 | 56/142

Fig. 72
Table of Shear Coefficients of w L at the Supports of Continuous Beams of Equal Spans Uniformly Loaded and Freely Supported.

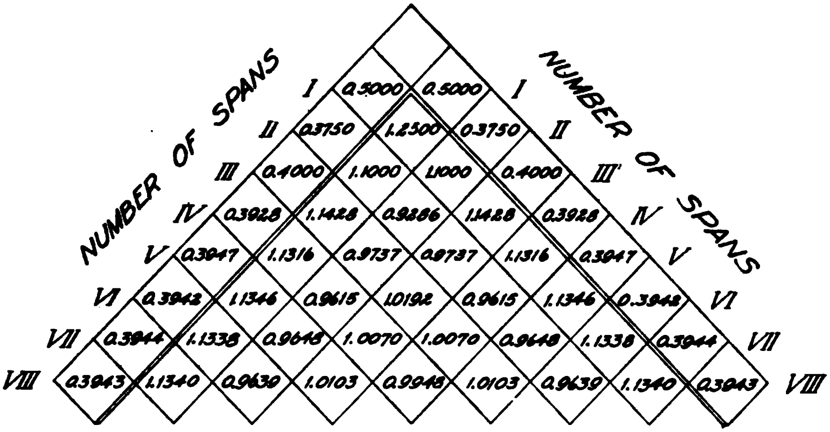
The coefficients along any sloping line in the diagram may be calculated from the two preceding coefficients in the same sloping line by the following rule: When the number of spans in the beam is *odd*, obtain the numerator of any coefficient by adding together the numerators of the two preceding fractions in the same sloping line, and then obtain the denominator likewise, by adding the two preceding denominators. When the number of spans is *even* obtain the numerator by adding twice the preceding numerator to once the numerator next before that, and the denominator likewise by adding twice the preceding denominator to the next preceding.

We have just calculated the shears in the cases of one and two spans respectively as given at the top of the diagram, and from them all the others can be readily calculated by the rule.

The reaction at any support is the numerical sum of the shears at the right and left of it, *i. e.*, where a shear at the left is marked by a prime,

$$R_n = S_n - S'_n$$

These reactions are to be obtained by taking the numerical sum of the shears at the right and left of each support. In a triangular table of reactions the same odd and even rule holds as for shears. But instead of writing the reaction coefficients as fractions with whole numbers for numerators and denominators, their values written for convenience as decimals have been tabulated in the accompanying diagram in which end reactions in any beam are identical with end shears.


NUMBER OF SPANS
NUMBER OF SPANS

I	0.5000	0.5000							I
II	0.3750	1.2500	0.3750						II
III	0.4000	1.1000	1.1000	0.4000					III
IV	0.3928	1.1428	0.9286	1.1428	0.3928				IV
V	0.3947	1.1316	0.9737	0.9737	1.1316	0.3947			V
VI	0.3942	1.1346	0.9615	1.0192	0.9615	1.1346	0.3942		VI
VII	0.3944	1.1338	0.9648	1.0070	1.0070	0.9648	1.1338	0.3944	VII
VIII	0.3943	1.1340	0.9639	1.0103	0.9948	1.0103	0.9639	1.1340	0.3943 VIII

Fig. 73. Table of Reaction Coefficients of wL for Continuous Beams of Equal Spans Uniformly Loaded and Freely Supported.

5. End Moments and Shears of Any Uniformly Loaded Span:

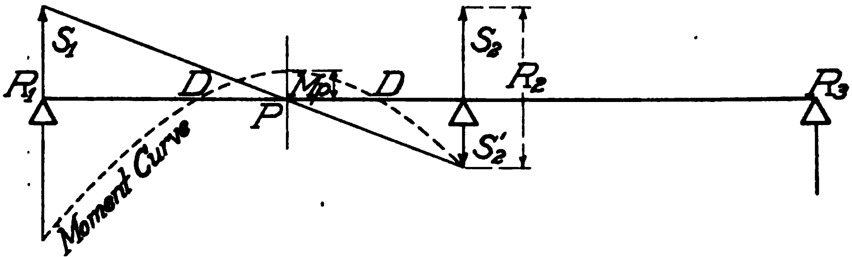


Fig. 74

Taking moments in the span about ends 1 and 2 successively, we have

$$M_1 = M_2 - S'_2L - \tfrac{1}{2}wL^2$$
$$M_2 = M_1 + S_1L - \tfrac{1}{2}wL^2$$

in which S_1 and S'_2 are intrinsically of opposite signs and S_1 is taken as positive and S'_2 as negative. Subtract and we have

$$M_2 = M_1 + \tfrac{1}{2}(S'_2 + S_1)L$$

i. e. if the span be taken as unity, the moment over any support taken as M_2 may be obtained from M_1 that at the support at the left of it by taking the numercial sum of M_1 and one half the numerical difference between the two end shears in that span when $S'_2 > S_1$, or taking the numerical difference between it and M_1 when $S'_2 < S_1$, in which due regard has been had for the intrinsic signs of M_1 and S'_2.

Again, in the next span at the right, we have

$$M_3 = M_2 + S_2L - \tfrac{1}{2}wL^2$$

and by the theorem of three moments

$$M_1 + 4M_2 + M_3 = -\tfrac{1}{2}wL^2$$

Let 1 be the free end, then, $M_1 = 0$, and by combining the foregoing equations to eliminate M_3 we find $M_2 = \frac{1}{5}S_2L$ from which it appears that when L is taken as unity the numerical value of the moment over the support next to the end is one fifth of the shear at the right of that support.

From this equation the moments over the supports next to the end may be obtained at once from the shear diagram.

For example, in a beam of two spans $S_2 = \frac{5}{8}$, when $L = 1$, hence $M_2 = -\frac{1}{8}$. In a beam of three equal spans $S_2 = \frac{5}{10}$, hence $M_2 = -\frac{1}{10}$ and in four spans $M_2 = -\frac{3}{28}$, etc.

Similarly at any support we find

$$M_n = -\tfrac{1}{5}(M_{n-1} + S_nL)$$

i. e. assuming $L = 1$ the moment at any support M_n may be calculated by taking one fifth the numerical difference of the shear at the right of it and the moment at the preceding support. It will, however, be more expeditious to calculate the successive values of the moments M_3, M_4, etc., beginning with M_2 as above by subtracting half the numerical difference of the shears from the preceding moment. The moment coefficients at supports in Fig. 77 may be thus calculated.

6. Law of Moment Magnitudes in Each Span of a Restrained Beam: Under section 2 (b) the summation moment curve was constructed first for a cantilever beam and then by moving the closing line AB of the moment curve downward to some new position A_oB_o, the + and — ordinates between it and the curve represented the moments under different symmetrical degrees of restraint at the end supports until when AB reaches $A''B''$ or the position of no restraint the ordinates represent the moments of a simply supported span which reach a magnitude of $wL^2/8$ at the center of span, in which w is the uniform load per unit length of span. This last position would correspond to the closing line *bd* in the accompanying Fig. 75 and to *OP* for the moment ordinate at mid span.

Suppose now the restraint at the support R_n is different from that at R_{n-1}. Let *ab* be the negative moment at R_{n-1} and *cd* that at R_n.

Then $$\frac{ab+cd}{2}=OO'$$

But $$OP=OO'+O'P$$

or $$OP=\frac{ab+cd}{2}+O'P$$

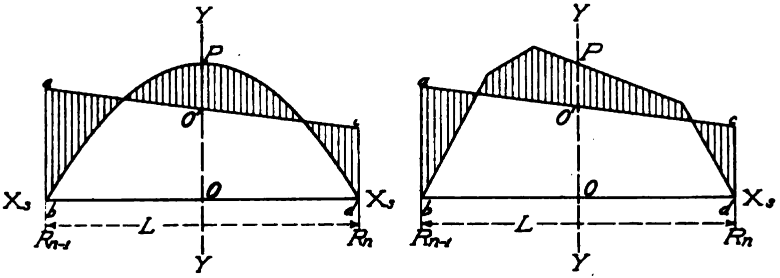


Fig. 75

Since no assumption has been made regarding the relative magnitude of *ab* and *cd* we have thus demonstrated the theorem that regardless of the distribution of the load and of the restraints at the supports we have: half the numerical sum of the moments over the supports plus that at mid span equals the moment of these loads at mid span when they are supported on a simple beam, which last moment in case of uniform loading is $wL^2/8$. This law holds good for the moment at mid span due to concentrated as well as to distributed loads and the demonstration will be the same in each case.

7. Center and End Moments in Continuous Beams Under Uniform Loads Resting on Equidistant Supports: In case of a uniform continuous beam of n equal spans of length L resting upon successive supports, R_1, R_2, $---R_n$, and supporting a uniformly distributed load of w per unit length of span, let M_1, M_2, $----M_n$ be the moments at the supports, C_1, C_2, etc., be the actual moments at mid span, and a the moment at mid span of a simple beam under the same load. Then the following equations express the theorem just proven:

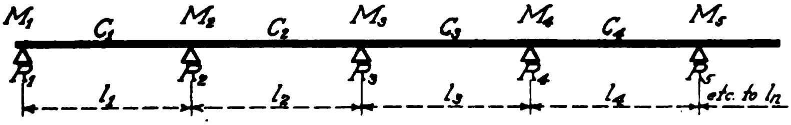


Fig. 76

(n = total number of spans uniformly loaded and $-a=-\frac{1}{8}wL^2$)

$$\frac{M_1+M_2}{2}-C_1=-a$$

$$\frac{M_2+M_3}{2}-C_2=-a$$

$$\frac{M_3+M_4}{2}-C_3=-a$$

$$\frac{M_4+M_5}{2}-C_4=-a$$

-----

and finally

$$\frac{M_n+M_{n+1}}{2}-C_n=-a$$

in which $a=wL^2/8$ and in which the values of the moments are treated algebraicaly, M being intrinsically negative and C positive.

The foregoing equations express the values of the center moments in terms of moments at the supports on either side of it. To obtain the moment at a support in terms of the center moments on either side of it, add two successive equations, viz.

$$M_{n-1}+2M_n+M_{n+1}=2(C_{n-1}+C_n-2a)$$

By the equation of three moments

$$M_{n-1}+4M_n+M_{n+1}=-4a$$

Take the difference, etc., hence

$$C_{n-1}+C_n=-M_n$$

i. e. the numerical value of the negative moment over a support can be obtained by adding the moments at middle of the spans on each side of it.

Again, let us substitute the values $-M_n=C_{n-1}+C_n$ and $-M_{n+1}=C_n+C_{n+1}$, in the final moment magnitude equation above and we find

$$C_{n-1}+4C_n+C_{n+1}=2a$$

which is the equation of relation of three successive center moments analogous to the equation previously given for three moments over successive supports. At an end of the beam, however, where we substitute $M_1=0$ and $-M_2=C_1+C_2$ in the first of the moment magnitude equation we get $3C_1 + C_2 = 2a$ for the three moment equation by which $C_2 = 2a - 3C_1$ may be computed from C_1, whatever be the number of spans in the beam.

In like manner we derive the corresponding equations for a continuous beam of any length of spans and any uniform loads which may be written as follows for the successive spans $n-1$, n and $n+1$,

$$M_{n+1}(L_n + L_{n+1}) + 2\,(C_nL_n + C_{n+1}L_{n+1}) = 0$$

and

$$\frac{C_{n-1}L_{n-1} + C_nL_n}{L_{n-1} + L_n} + C_n + \frac{C_nL_n + C_{n+1}L_{n+1}}{L_n + L_{n+1}} = a_n$$

which last equation reduces for the two end spans to the form:

$$C_1 + \frac{C_1L_1 + C_2L_2}{L_1 + L_2} = a_1$$

It may be observed that the moment magnitude equations also apply to any continuous beam under any loading whatever concentrated or distributed in any manner or with spans of unequal length, and different degree of restraint over the supports, except that the value of a in these equations would have to be that for the applied moments for each respective span regarded as a simple beam. Thus the equation of moment magnitudes combined with the three moment equation of the mid span moment furnishes in a simplified form the necessary conditions for the solution of any problem presented by a series of continuous spans under different uniform loads.

By adding the equations of center moments given above and observing that moments at equal distances from the ends are equal, we have the following equations respectively, for beams of I, II, III or more spans.

I $C_1 = a$
II $4C_1 = 2a$
III $4C_1 + 3C_2 = 3a$
IV $4C_1 + 6C_2 = 4a$
V $4C_1 + 6C_2 + 3C_3 = 5a$
VI $4C_1 + 6C_2 + 6C_3 = 6a$
VII $4C_1 + 6C_2 + 6C_3 + 3C_4 = 7a$
– – – – – – – –
N $4C_1 + 6C_2 + \ldots + 6C_{n/2} = na$
$N+1$ $4C_1 + .6C_2 + \ldots\ 6C_{n/2} + 3C_{n/2} = (n+1)a$

from which it appears that there is a certain law of coefficients for an even number of spans and a certain different law for an odd number of spans.

Now it may be shown in general that

$$
\begin{aligned}
C_2 &= 2a - 3C_1\\
C_3 &= -6a + 11C_1\\
C_4 &= 24a - 41C_1\\
C_5 &= -88a + 15C_1
\end{aligned}
$$

etc.

These last equations which hold for beams of any number of spans whatever enable us to determine the values of the center moments in a beam of any given number of spans by substituting them in the equation given above for a beam of that given number of spans. We thus obtain the following values for beams of a given number of spans:

I $C_1=a$
II $C_1=\frac{1}{2}a$
III $C_1=3a/5$ $C_2=a/5$
IV $C_1=4a/7$ $C_2=2a/7$
V $C_1=11a/19$ $C_2=5a/19$ $C_3=7a/19$ etc.

We have already shown that $-M_{n+1}=C_n+C_{n+1}$ from which we find the moments at the supports for a given number of spans.

II $-M_2=a,$
III $-M_2=4/5\ a,$
IV $-M_2=6/7\ a,$ $-M_3=4/7\ a,$
V $-M_2=16/19\ a,$ $-M_3=12/19\ a,$ etc.

But C_1 may be also computed in many other ways for example from S_2 as follows: $C_1=a+\frac{1}{2}M_2$ by the first of the moment magnitude equations, and since $M_2=-1/5\ S_2L$, as above, we have $C_1=a-S_2L/10$.

Each of the moments at the supports and at mid span is the product of wL^2 by a fractional multiplier or coefficient, which coefficient expresses the value of the moment when w and L are each taken as unity. By using the methods already developed the values of the moments at mid span and at the supports may be calculated by either of several different ways. A tabular diagram both of the center moments and the moments at the supports when w and L are taken as unity is here given.

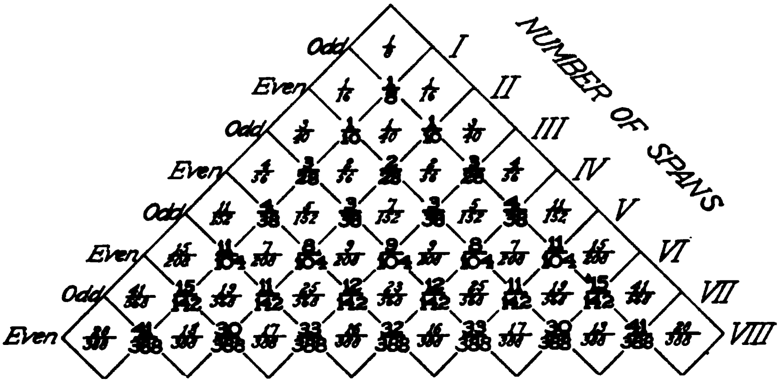

NUMBER OF SPANS
Odd 1/8 I
Even 1/16 1/8 1/16 II
Odd 2/40 1/10 1/40 1/10 2/40 III
Even 4/56 3/28 2/56 2/28 2/56 3/28 4/56 IV
Odd 11/152 4/38 5/152 3/38 7/152 3/38 5/152 4/38 11/152 V
Even 15/208 11/104 7/208 8/104 9/208 8/104 9/208 8/104 7/208 11/104 15/208 VI
Odd 41/568 15/142 13/568 11/142 25/568 12/142 23/568 12/142 25/568 11/142 13/568 15/142 41/568 VII
Even 28/388 41/388 13/388 30/388 17/388 33/388 16/388 32/388 16/388 33/388 17/388 30/388 13/388 41/388 28/388 VIII

Fig. 77
Table of Coefficients of Center Moments and Moments Over Supports of Continuous Beams of Equal Spans Uniformly Loaded and Freely Supported.

This table begins at the top with the moment coefficients already obtained for beams of several spans, which furnish a sufficient and most convenient basis for calculating the entire table according to the odd and even rule about to be stated. Care must be taken, however, not to reduce any of these fractions to lower terms by cancelling common factors in any numerator and denominator, as has inadvertently been done in the center moments of the beam with VIII spans, which should have 776 in each denominator, and each numerator doubled. The moments over the supports, however, are in correct form. The center moments are all positive, the others negative. The table is checked by adding two successive center moments to obtain the moment at the intermediate support.

In this table any *center* moment coefficient may be obtained from the two preceding coefficients on the same sloping line by the following odd and even rule.

In case a beam of an *even* number of spans, add the two preceding numerators to form the required numerator and the two preceding denominators for the required denominator, but in case of an *odd* number of spans, add twice the preceding number to once the next preceding to obtain the required numerator or denominator.

In order to obtain the odd and even rule for coefficients of moments at the supports replace the word *spans* in the above rule by *supports*.

If a continuous beam be completely fixed at its two end supports, then the tangent to the deflection curve is horizontal over every support and every center moment is $wL^2/24$, while every moment at a support is $wL^2/12$.

Now if a table be constructed that expresses the amount of change which would occur in releasing the restraint by freeing the ends, the changes of these moments would be obtained by taking the differences $wL^2/24 - C_n$ at mid span, and $wL^2/12 - M_n$ at the supports. Such a table containing the differences both at mid span and supports follows the odd and even rule for the moments over supports, and the word *support* must be put in place of *span* in the original rule to make it apply to such a table. In this table it is important that the correct sign, positive or negative be affixed to the differences placed in the table, and that the denominators 12, 48 and 60 be used in the cases of beams with 1, 2 and 3 spans respectively.

The tables of reactions and shears at supports follows the same odd and even rule as moments at supports, but if the fractions representing the coefficients for shears, reactions and moments at supports for each beam be written with the same denominators as are used in the table for center moments, the odd and even rule in all these tables will then be the same as for center moments.

As previously shown there must be separate rules for calculating the series of center moments for beams of an odd number of spans from those of an even number of spans. It is apparent from observation of the table that the odd and even rule stated is correct. But an odd and even rule for center moments must involve an odd and even rule for the moments at supports also, for if these latter were obtained from the equation $-M_n = C_{n-1} + C_n$ by simply adding the numerators of the center moments without altering the denominators, the same odd and even rule would be found for the moments at the supports as for center moments. But the rule for the coefficients of the pier moments M_1, M_2, etc., is the reverse of that for the mid span moments C_2, C_3, etc., in the table because these coefficients of M are reduced by cancelling a common factor 2 from each of them in every odd number of spans, and the factor 4 from them in every even number of spans.

For convenience of calculation and comparison the following table of decimal values of the moment coefficients of the negative moments at the supports is also inserted from which it will be seen how rapidly the moments at any given support approach a limiting value as the number of spans increases. For example, M_2 approaches .106 due to the relation $M_2 = -S_2/5$, previously obtained, in which S_2 approaches .53, as will be seen from the table of shear coefficients.

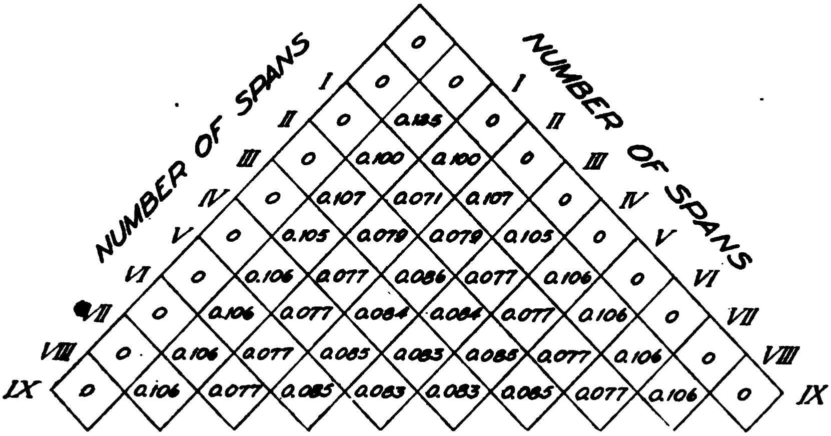

NUMBER OF SPANS
NUMBER OF SPANS
0
I 0 0 I
II 0 0.125 0 II
III 0 0.100 0.100 0 III
IV 0 0.107 0.071 0.107 0 IV
V 0 0.105 0.079 0.079 0.105 0 V
VI 0 0.106 0.077 0.086 0.077 0.106 0 VI
VII 0 0.106 0.077 0.084 0.084 0.077 0.106 0 VII
VIII 0 0.106 0.077 0.085 0.083 0.085 0.077 0.106 0 VIII
IX 0 0.106 0.077 0.085 0.083 0.083 0.085 0.077 0.106 0 IX

Fig. 78. Table of Coefficients of Moments over the Supports Beams Freely Supported.

There are many remarkable relations between the numerators and denominators in the tables of shears and moments. For example, while the denominators of the moment coefficients at the supports in any given number of spans n is the same as those of the shears for the same number of spans n, the numerator of M_2 is the same as the numerator of S_1 for $n-2$ spans. Other relations exist which seem equally remarkable.

8. End Shears and Maximum Moments in the Several Spans of a Continuous Beam Uniformly Loaded and Resting on Equidistant Supports: By equation (2) section 2 (b) it is evident that wherever the moment reaches a maximum value the shear vanishes because the condition for a maximum value of M is that its differential coefficient vanish. Consequently the maximum value of the moment in the span R_1R_2 Fig. 74 is found at P where the shear vanishes. But by similarity of triangles

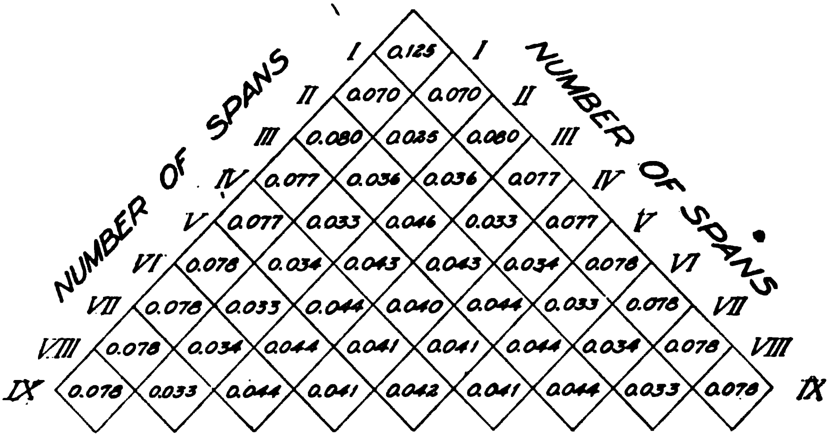
$$\frac{S_1}{PR_1}=\frac{-S'_2}{PR_2}=\frac{S_1-S'_2}{PR_1+PR_2}=\frac{wL}{L}=w$$

Hence, $PR_1=S_1/w$ and $PR_2=-S'_2/w$

Now M_p the moment at P may be calculated as follows:
$M_p=M_1+S_1(PR_1)-\frac{1}{2}w(PR_1)^2=M_1+S_1^2/w-\frac{1}{2}S_1^2/w=M_1+\frac{1}{2}S_1^2/w$
and similarly

$$M_p=M_2+\tfrac{1}{2}S_2'^2/w$$

in which M_1 and M_2, etc., are negative. These formulas permit the ready calculation of the maximum moment coefficients from the preceding tables of moments and shears at supports by assuming $w=1$.


NUMBER OF SPANS
NUMBER OF SPANS

I	0.125								
II	0.070	0.070							
III	0.080	0.025	0.080						
IV	0.077	0.036	0.036	0.077					
V	0.077	0.033	0.046	0.033	0.077				
VI	0.078	0.034	0.043	0.043	0.034	0.078			
VII	0.078	0.033	0.044	0.040	0.044	0.033	0.078		
VIII	0.078	0.034	0.044	0.041	0.041	0.044	0.034	0.078	
IX	0.078	0.033	0.044	0.041	0.042	0.041	0.044	0.033	0.078

Fig. 79. Table of Coefficients of Maximum Positive moments between Supports for Continuous Beams Freely Supported.

The accompanying table of these coefficients is expressed in decimals, because owing to the fact that they depend on the squares of end shears their values when expressed in common fractions soon require inconveniently large numbers. It will be seen that by reason of the fact that the value of S_1 next the free end rapidly approaches the value of 0.4 as the number of spans increases we find that the value of the coefficient of the maximum moment in the end span $= \frac{1}{2}S_1^2$ rapidly approaches the value .08 as the number of spans increases while that in the center span approaches .04, but not so rapidly.

Apply the preceding equation to obtain the maximum moment coefficient in case of two equal spans

$$M_p = \tfrac{1}{2}(3/8)^2 wL^2 = \frac{9wL^2}{128}$$

The maximum moment in an end span of three equal spans is

$$M_p = \tfrac{1}{2}\,(2/5)^2 wL^2 = \frac{2wL^2}{25}$$

and in the middle span it is

$$M_p = -\frac{wL^2}{10} + \tfrac{1}{2}(1/2)^2 wL^2 = \frac{wL^2}{40}$$

Similar calculations may be made for the coefficients in beams of any greater number of spans.

The distance between points of zero bending moment in a continuous beam, which points are identical with the points of inflection is readily found from the maximum moment because that part of the span which reaches from one point of inflection to another, which part we may designate as $DD=L'$ in Fig. 74 is a simple beam supported by the shears at D and D in which the maximum moment at P is $\frac{1}{8}wL'^2=M_p$. Hence $L'=\sqrt{8M_p/w}$ and $R_1D=R_1P-DP$, $R_2D=R_2P-DP$ in which $DP=\frac{1}{2}L'=\sqrt{2M_p/w}$.

From these equations the distances of the points of inflection from the supports may be readily calculated.

9. Position of Inflection Points and Distribution of Bending Moments in Restrained Beams as Affected by the Moments of Intertia at Mid Span and at Supports. Assume that both ends of a span of effective length $L=2x_2$ between supports are perfectly restrained, or built in horizontally and uniformly loaded so that the suspended span as shown in Fig. 65a has a length of $2x_1$ between the points of zero bending moment which are also the points of inflection. Let I_1 be the moment of inertia thruout the suspended span and I_2 that in the segments of the span between the supports and the suspended span. Then the ratio of $i=I_1/I_2$ is connected with the position of the points of inflection and the subdivision of the total applied bending moment between the moment $C=y_1$ at mid span and the moments $M=y_2-y_1$ at the supports in a manner that will be now investigated.

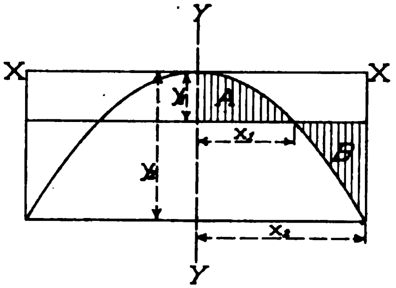


Fig. 65a

In order that the bending due to the uniform loading may be symmetrical the tangent to the deflection curve must be horizontal at mid span as well as at the ends, *i. e.* whatever amount of negative bending is produced by the total negative bending moment area B must be balanced by an equal amount of positive bending due to the total positive bending moment area A. Now the amount of actual bending produced by a moment area is directly proportional to that area and inversely proportional to its stiffness as measured by its moment of inertia. Hence we must have

$$A/I_1=-B/I_2 \text{ or } A=-iB$$

which equation may be written

$2x_1y_1/3 = i\ (x_2y_2 - x_1y_1)/3 - i(x_2 - x_1)y_1$. But $\frac{y_2}{y_1} = \frac{(x_2)^2}{x_1}$

Hence $i = \frac{2x_1/x_2}{2x_1/x_2 + (x_2/x_1)^2 - 3}$

from which the ratio of the moment of inertia at mid span to that at supports can be calculated in order to make the point of inflection fall at a fractional part of the span from the center $= x_1/2x_2$. Values so calculated are given in the accompanying table.

x_1/L	.10	.15	.20	.25	.3	.35	.4	.45	.50
$i = I_1/I_2$	.02	.07	.20	.50	1.20	3.20	10.00	53.00	∞
C/a	.04	.09	.16	.25	.36	.49	.64	.81	1.0
M/a	.96	.91	.84	.75	.64	.51	.36	.19	.00
C/M	.04	.10	.20	.33	.56	.96	1.8	4.2	∞

in which $L = 2x_2$, and $a = WL/8 = \frac{1}{2}wx_2^2$.

The bending moment at mid span is $C = \frac{1}{2}wx_1^2$ and that at the supports is $M = \frac{1}{2}w(x_2^2 - x_1^2)$

The ratios of these moments to the total moment in the span, *i. e.* to $\frac{1}{2}wx_2^2 = wL^2/8$, and the ratio C/M have been also calculated and are likewise inserted in the table.

Now starting with any given or assumed value of either of these ratios the table shows the corresponding values of the other ratios, For example, starting with i given, we find the position $x_1/2x_2$ of the corresponding points of inflection and the ratio C/M of the center moment to the moment at the supports in such a restrained span.

It will not practically interfere with the accuracy of the values given in this table to have the point of the span where I_1 changes to I_2 occur at some little distance either side of the point of inflection because near that point the bending is very small under any circumstances owing to the smallness of the bending moment in that vicinity, a fact which renders the moment of inertia in that part of the span of little importance.

From the foregoing investigation it is clear that the level y_1 of the closing line that divides the total applied moment area of the load into the positive and negative moment areas A and B such that $C + M = y_2$ is fixed in this case of restrained ends by the ratio i of the moment of inertia at mid span and supports, the case when the moment of inertia is uniform thruout the span or $i = 1$, $x_1/x_2 = .583$ and $C/M = .5$ having been previously considered at length. The magnitude of the total moment $C + M = \frac{1}{2}wx_2^2$ in the span $L = 2x_2$

is known in case of a beam to be the same as the statical moment of the load but in case of a plate or slab it has been previously shown that the total applied bending moment is largely reduced in types III and IV by reason of twisting moments.

Moreover, the load that produces the bending moment in the panel is not uniformly distributed along the span but increases uniformly from support to mid span. The present investigation of points of inflection etc., of beams consequently cannot be applied to the panels of a continuous slab. A separate investigation will be required for them.

It will be further noticed that any increase of I_2 relatively to I_1, *i. e.*, any decrease of i such as takes place by placing more steel over the supports than elsewhere, tends to move the point of inflection toward mid span, and to decrease C/M. For example, if $i = .5$, then $x_1/x_2 = .5$ and $C/M = .33$ or the moment at the supports is three times that at mid span.

The position of the points of inflection is also affected by other circumstances than the ratio i of the moments of inertia among the most effective of which is the ratio $N = L/d$ of span to depth and the consequent relative amounts of beam and arch action. The preceding investigation holds without modification in cases of long spans only, *i. e.* where the span is at least 25 times the thickness arch action becomes negligible, being in such cases not more than 5 percent.

10. Shear in Continuous Beams. Because the horizontal shearing deformation is zero where the vertical shear is the greatest the safe shearing resistance of the concrete at the support of a continuous or restrained beam is twice as great as for a simple beam and may be figured as such using the ratio of depth to span N in determining the shear value as though it were a simple beam and doubling that value, see curve diagram Fig. 50, Chapter VI. The reinforcement at the support adds to the safe shearing value an amount which may be taken as 10,000 lbs. per square inch for the steel in the tensile zone at the top of the beam and half that amount for the bottom flange steel.

From the foregoing discussion it is evident that the horizontal shear or bond stress in the continuous beam vanishes at the supports, where the Joint Committee rule based upon the common theory of flexure applicable to simple beams would make it a maximum. In the vicinity of the line of inflection bond stress is a maximum and is about half as great as it would be at the end of a simple beam of the same depth carrying the same load.

The reinforcement to resist the diagonal tension of shear may be most advantageously placed in the continuous beam in such wise that it traverses the web from the zone of tension in the bottom to the zone of tension in the top so that steel resists tension throughout its entire length relieving the web of the major portion of the diagonal tension at that particular section where it is a maximum.

The manner in which the reinforcement in the tension zone in the suspended span operates in conjunction with the tensile reinforcement in the cantilever portion of a continuous beam may be illustrated by considering first a case in which these reinforcements consist of straight rods lying in the outer portion, top and bottom respectively, of the upper and lower tension zones. Such reinforcing elements would be lapped past the line of inflection toward the support in the bottom part of the beam and toward the center of span in the top. The bond stress would be a maximum at the line of inflection and would decrease each way therefrom in both upper and lower sets of bars. Such distribution of bond shear would produce compressions and tensions crossing each other at the line of inflection, wherein it would differ from a homogeneous beam.

Consider the case now where the steel is bent diagonally downward through the line of inflection and forms the reinforcement not only in one continuous piece in the cantilever but in the suspended span portion as well. That portion of the tension in the steel at the line of inflection which is not equalized by the bond shear stress occurring between it and mid span would be transferred as steel tension into that part of the bar in the tensile zone of the cantilever beyond, and the state of stress in the bar would be that of tension throughout its whole length. The bond stress in the concrete would be reduced by the total amount of tension transmitted across the section at the line of inflection. Under this distribution of stress there would be an overlapping of compression in the concrete above the line of inflection toward mid span and a corresponding overlapping past the line of inflection toward the support by compression in the concrete below, a phenomenon which, from extensometer measurements of the concrete stresses at and near the line of inflection, has been mistaken for arch action.

Thus the suspended span portion differs somewhat in its operation from that of a simple beam freely supported. It is less subject to cracking of the concrete by bond stress which is considerably reduced at the end of the suspended span portion by the tension in the steel. Ordinarily checking would first occur in the cantilever portion of the continuous beam were it not for the fact that the upper part of the beam is usually monolithic with the floor slab thus providing an excess section.

Hence a continuous beam is a much more satisfactory member than a simple beam first because it is much stiffer and the concrete is less liable to crack because of this increased stiffness and second on account of the lower relative value of the horizontal shearing stress which determines such checking.

10. Economic Design of Beams. The moment coefficient for bending of a continuous beam is one third as great at mid span as in case of a simple beam and two thirds as much at a support. For safety, ample lap of the bars is needed. Hence by carrying a part of the reinforcing rods required at mid span up over the support from near the point of inflection or so far that the negative moment in the case of a single panel load will be taken care of by slab reinforcement parallel to the beam, we need, considering moment, only two thirds the section of the steel for about one third the length and one third of the section of metal for two thirds the length of the continuous beam that is required for the simple beam. Thus the metal required for the continuous beam action is one third that required for the simple beam of the same depth and moreover the continuous beam type is safer to erect since the work is more securely tied together and with good concrete it will not fail suddenly but only by stretching out of the metal to the point of ultimate fracture in case it is loaded with three or four times the load it was calculated to sustain.

Evidently less steel would be required to carry any given load the greater the beam depth. Usually the depth to be used in an ordinary building is determined from the standpoint of appearance and of the extra cost of walls for a given clear story height rather than from the theoretical economical proportions of steel and concrete. A mistake too frequently made is that of proportioning the beams too narrow and deep, particularly where they are closely spaced. Such construction is lacking in resistance to high temperature because too great an area is exposed and should be avoided on that account. Ten or twelve inches should be the minimum width for reinforced concrete beams in warehouse buildings that are intended to be fire resisting to a high degree and such a width for moderate spans of 16 to 18 feet gives ample concrete to properly surround the reinforcement and furnish the necessary resistance to shear. In general there should not be less than 1″ of concrete between the bars or a width not less than one and one fourth times the diameter of the bars if the bars are parallel for any considerable length. Where bond shear is relatively small and the metal is concentrated as commonly occurs at the top of a continuous beam and there is ample lap beyond the support this requirement becomes of no especial importance.

Relative to the economic proportion of concrete and steel, the general relation to be observed, is that the amount of steel decreases with the depth of the beam, while the amount of the concrete increases. With one percent of steel as the element of reinforcement, it is evident that the concrete element will cost more than the steel element on the basis of five dollars a yard for the concrete and steel at fifty dollars per ton. Hence with ordinary values of steel and concrete, the limiting permissible percentages and relation of safe working stresses fix economic proportions. In the continuous beam, however, where there is double reinforcement over the support, a closer approximation to a balance between the cost of concrete and metal may be approached than in the simple beam.

However, in the T-beam which is the usual construction in buildings this percentage would be based upon the area of the beam below the slab plus the area of that part of the slab above and on each side of the rib which it is permissible to consider as forming part of the compression flange of the beam, and the economic proportions would have a smaller proportion of steel, since the portions of the slab figured in with the beam are not added material as far as the beam is concerned, and hence the comparison should be based more properly upon the area of the rib added below the slab to form the beam. Hence the economic proportion of steel would in general be reduced below that of the limiting proportion fixed to secure conservative working stresses in the concrete.

This conclusion, that the cost of the steel in the T-beam should be less than the concrete is strengthened by the consideration that the cost of the centering increases with increase of depth of the beam. These practical considerations seem to have been entirely overlooked in the discussion of the paper presented to the American Society of Civil Engineers in 1906 by Capt. Sewell, on the subject of economic construction of reinforced concrete floors.

No mathematical formula has been devised which will take into consideration all of the variable elements of cost. Trial designs and the practical judgment of the constructor enable him to find an approximate and satisfactory solution of this complex problem.

The intimate relation of horizontal shear to permissible percentage of steel which we have pointed out earlier in the discussion is but another of the complex elements entering into the problem.

11. **Summary.** In this chapter the graphical relations between loads, shears, bending moments, slopes and deflections are developed in a novel manner that admits of a ready comparison of these magnitudes.

From this development Clapeyron's Equation of moments at three successive supports of a continuous beam follows at once as well as the values of the shears at the supports.

The fundamental equation between the moment magnitudes at mid span and at supports in any span is here obtained for the first time in its general form, and from it the general equation is deduced between three successive mid span or center moments, heretofore unknown.

These equations show that the rule for the calculation of center moments for beams with an odd number of spans follows a different law from that of the center moments of a beam with an even number of spans and the relation of the odd and even rule for moments and shears at supports is derived from that of center moments.

The amount and position of the maximum moment in any span of a continuous beam is likewise obtained as well as the position of the points of inflection.

The characteristic action of reinforcing rods extending from tension zone in the bottom at mid span to tension zone in the top of a beam at the supports is developed. It consists in tension in the steel at the inflection points as well as elsewhere, and compression in the concrete at the same points.

Because the deflection of a fully restrained beam is but one fifth of that of the simply supported beam and the radius of curvature determining the angular shear deformation that produces cracking of the concrete is reduced with the increased radius of curvature in the continuous beam, steel stresses three to four times as great may occur in fully restrained beams before the concrete checks as is possible in case of simple supported beams.

CHAPTER VIII

INDIRECT STRESSES GENERATED BY BOND SHEAR AND THE LAWS OF DISTRIBUTION GOVERNING THESE MECHANICAL ACTIONS

1. Bond Shear in Blocks. Let *R* Fig. 80 represent a cylindrical metallic rod embedded in a block of concrete *B B* in which it has been cast. Let it rest on a ring shaped support *SS* having a central aperture somewhat larger than the rod. Then the weight *W* applied to the rod will induce a bond shear at the surface of the rod, whose intensity will depend upon several factors, among which will be the magnitude of the weight *W*, the diameter *D* of the rod, the length *L* of the embedment, and the distance of the point considered from the beginning of the embedment. Other things being equal, the intensity of the bond shear will be greatest at the point of embedment nearest the point of application of *W*, whether the rod be in tension or in compression. Since the concrete which surrounds the rod is in a state of shearing stress along the surface of the rod, it is in a state of stress that may also be defined otherwise by saying that there is indirect or induced tension in the concrete along lines sloping at 45° towards the axis of the rod, represented on Fig. 80 by the arrows with heads at the surface of the rod, and also compression induced in the concrete along lines sloping at 45° away from the axis, the intensity of each of these induced stresses at the surface of the rod being the same as that of the bond shear at the surface and growing less at greater distances from the axis nearly inversely as the distance.

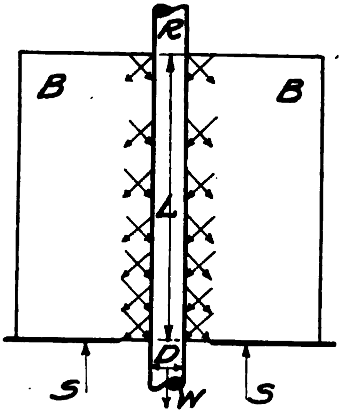


Fig. 80.

The arrows may be regarded as representing lines of force or stress in the concrete in such a manner that the intensity of the stress is proportional to their nearness to each other. The total bond shear between rod and concrete amounts to *W*, and is so distributed on the surface of the rod that it is small at the point of embedment most distant from *W* and is greatest at the points nearest to *W*. The bond shear at any point is the increment of the tension in the rod at that point and there is no element of tension in the rod which is not balanced and transmitted to the concrete by a corresponding equal element of bond shear.

The block tends to decrease in length by reason of the load thus transmitted to it at the same time that the part of the rod embedded in the concrete tends to increase in length, so that the point where the first slipping will occur is at the point nearest to *W*. But if no actual slipping occurs the distortion is greatest at this point. This is the reason that the greatest intensity of bond shear occurs at this point.

In the space where the arrows are represented it might at first be thought that the radial components of the induced tensions and compressions would neutralize each other. But such is not the case because it is a fundamental property of stress in any material that a shear in any plane (as for example the vertical plane in this case) in order to maintain equilibrium is necessarily accompanied by an equal opposite shear on a plane at right angles with it, so that these induced vertical stresses act to produce radial shears of equal intensity on each horizontal plane.

This horizontal shear in the block diminishes somewhat as the radius increases but does not vanish until we reach the inner edge of the ring *SS*. Beyond the inner edge of *SS* a vertical compression exists in the block which causes a pressure upon the ring *SS*, whose intensity is greatest at the inner edge of *SS*, but which diminishes beyond the inner edge of *SS* in a ratio which need not here be considered.

The matter of importance in this discussion is the manner in which the stress is transmitted thru the concrete around the rod, where the concrete has lines of stress in it similar to those in the web of a built beam which cause the two flanges to coact as do the rod and exterior part of the block in this case.

The induced tensions which have just been described are the same as those frequently designated as diagonal tensions. Failure arising from excessive stress of this kind would evidently give rise to cracks nearly at right angles to the arrows representing the induced tensions, i. e. to cracks along the lines of compression, but such cracks could not occur without excessive elongation along the rod, nor would they be expected to occur in a block at all under ordinary circumstances, because they would be preceded by slipping of the bond, after which the diagonal shear would be relieved where the slipping took place and transferred to some point further along the rod.

2. Bond Shear in Splices. Let Fig. 81 represent a splice in a belt of reinforcing rods with laps of length L. The entire effectiveness of the splice depends upon the bond shear at the surfaces of the belt rods in which the action of the embedment is in many particulars like that already discussed, but has several new features especially by reason of the dissymmetry of the embedment laterally, altho longitudinally it has perfect symmetry such as the block did not possess.

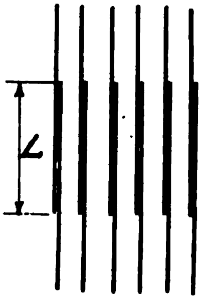


Fig. 81

The lateral arrangement is such that the stresses in the concrete between the rods on the sides where they are nearest together is of much greater intensity than elsewhere, so that the diagram of the stresses about a rod would be much like that shown in Fig. 80, except that on one side of the rod the arrows would cover the space very closely, while on the other side the arrows would be few and far between. Moreover the successive arrows would be just as near each other at one end of L as at the other end, but nearer together at each end than nearer the middle of L. The bond shear would fail first on each rod at that end of L furthest from the end of the rod. The lines of force in the horizontal plane of the belt would be as just described, but lines of force which start out from the rods in a direction above or below that plane would curve around spirally from one rod to the other, for each line of force must have one end on one rod and the other on some other rod near by.

The strength of the splice depends upon the bond shear at the surfaces of the rods, and on resistance to the induced tensions and compressions along the lines of stress afforded by the concrete. The integrity of the concrete embedment under the action of these stresses is essential to the transmission of force along the splice, and were the limiting values of either bond shear, indirect tension, or compression, exceeded in the concrete the splice would fail.

3. **Bond Shear in Beams.** Let Fig. 82 represent a reinforced concrete beam of length L supported at the ends and loaded with two equal weights W each placed symmetrically at the same distance, nL from the end nearest to it. Then the total vertical shear at any point between the equal weights is zero and the bending moment is nWL, while at any point between either load W and the end nearest to it the total vertical shear is W and the moment is Wx, where x is the distance of the point from the end nearest to it. In such a beam where there is no vertical shear at any point of the middle segment there is no horizontal shearing stress at any point of this segment, consequently the bond shear stress is zero thruout this segment.

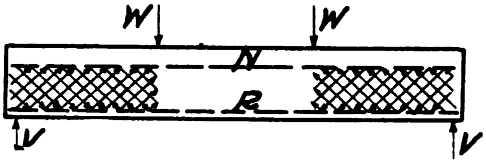


Fig. 82.

Let N represent the position of the neutral axis. The concrete above N is in direct compression and that below N is subject to direct tension along horizontal lines none of which are represented in Fig. 82. The entire length of the reinforcing rod R is under tension, the central segment between the applied weights being under a uniform tension thruout which will be determined ultimately by the bending moment WL, for if the weights W are sufficiently great or other contingencies such as rapid setting, or long continued loading have occurred, enough vertical cracks will have developed at the bottom of the beam in this segment to have practically destroyed its direct tensile resistance to horizontal force. Provided there is sufficient horizontal reinforcement to safely resist this horizontal tension and the concrete above *N* is sufficient to resist the compression the existence of vertical cracks in this segment of the beam at the bottom is not to be regarded as a sign of structural weakness or ultimate failure of the beam. These may under these conditions be regarded as harmless characteristic cracks. But in case the reinforcement is insufficient, yielding will occur in this part of the beam where the moment is greatest and such yielding will cause failure.

In the end segments of the beam the moment of the bond shears *S* per unit of length of reinforcement represented by *Sjd* must be equal to the increment of the bending moment, *i. e.* equal to the total vertical shear *W* provided it be assumed that the reinforcement furnishes the entire direct tensile resistance and the concrete none. Assuming that case for safe design, the bond shear is represented in Fig. 82, in the same manner as in Fig. 80, viz: the arrows with heads at the reinforcing rods represent the lines of indirect tension in the concrete, and those with heads at the neutral axis *N* represent the lines of compression induced by the bond shear. The concrete is abundantly able to afford the necessary resistance to the indirect compression, but may be entirely unable to afford the necessary resistance to the indirect or diagonal tension, which may consequently cause cracks perpendicular to its direction, that is along the lines of indirect compression, and at the same time produce rupture of the bond shear. In beams so heavily reinforced as to prevent central yielding, failure due to rupture of diagonal tension and rupture of bond along the steel ordinarily occurs under overload. These two accompany each other because they are of practically equal intensity as noticed before, and the cracks due to slipping of the steel are continuous with those due to diagonal tension. Additional reinforcement to resist diagonal tension consists of stirrups and the like, which are enabled to do this by their own secondary system of bond shears and anchorage.

The system of induced or diagonal tensions which have just been treated are necessarily modified somewhat in amount and direction by their combination with the direct horizontal tensions in the concrete due to the bending where these direct tensions are not eliminated by vertical cracks. But the existence and effectiveness of the induced tensions depending on the bond shear remains practically unimpaired so long as the bond is intact.

It should be noticed that the distribution of the bond shear on the surface of the reinforcement must be more intense on the upper sides of the rods by reason of the greater rigidity of the concrete on that side, due to its backing above, and lack of backing below.

In case of a beam carrying a load distributed differently from that assumed in Fig. 82, there may be no segment of any finite length where the shear vanishes, but such a segment of infinitesimal length will exist wherever the bending moment is constant, *i. e.* wherever it is a maximum or a minimum. Moreover the total values of the bond shear per unit of length of the reinforcement, multiplied by *jd* will equal the total vertical shear in case the reinforcement takes the entire direct tension. In such a case the characteristics of the middle and end segments of Fig. 82, merge into each other somewhat, in a manner not difficult to understand.

It will be readily seen that the manner in which the stresses are distributed in any case of reinforcement depends entirely upon the applied forces and the distribution of the rigidities due to the amount and distribution of the steel, but that the coaction of concrete and steel is brought about by stresses communicated from one to the other thru the medium of bond shear which is vital to the discussion of any such question, and that together the steel and concrete form a combination or machine which has properties as a whole which are different from the properties of either of the constituents. What those properties may be in any case must be determined from a careful analysis of the particular case under consideration.

It should be further noticed that the lines of tension that originate or are generated in the bond shear at any given element of the surface of a rod are to be thought of as physically independent of the lines of compression originating at the same point in such fashion that the tension starting at a given point *a* say, is held in equilibrium at the other end of that line by a tension at some point *b*, while the compression which originates at the same point *a*, is resisted and held in equilibrium by a resistance to compression at some entirely different point *c*. Moreover, it may be that *b* and *c* are on the surfaces of entirely different bodies, as frequently occurs in case of splices like Fig. 81.

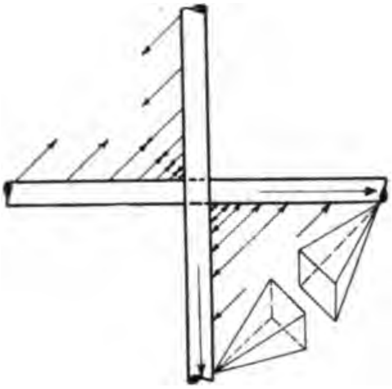


Fig. 83.

4. **Bond Shear in Slabs.** Let Fig. 83 represent the crossing of two reinforcing rods under tension in a multiple way reinforcement as for example in a slab where the shearing stress of the bond resists any tendency of the rods to slip longitudinally in the direction of the long arrows. Then the mutual action of the lines of tension and compression arising from the bond shear is that represented in the figure where the arrows with points against the rods represent lines of tension and those pointed away from the rods lines of compression. There exist other lines due to the bond shears besides those here represented which are similar in their disposition and action to those found in beam action which has been already discussed. But such other lines are here omitted from discussion because by the principle of rigidities the lines which are here represented as affording short and direct connections thru the concrete are necessarily the ones which include the predominant action of the bond shear. This predominant action is here seen to resemble in its general nature that of a splice, but has laws of distribution peculiarly its own by reason of the relative situation of the rods in the concrete, which is such that the intensity of the stress developed is greatly intensified in the immediate vicinity of the point of contact of the rods.

Since in any structure where there are several parts of the structure, each of which is to carry its part of the load the question as to how the load will be divided between them is determined by their relative rigidity or stiffness, that part which lacks stiffness and yields easily will carry little of the load and will leave the brunt of the work to those parts that are rigid and unyielding. Furthermore, since the deformation multiplied by the mean force (or half the final load) is the work done during deformation, it is evident that for a given applied load the more rigid parts carry their part of the load with less deformations than the other parts could carry it. Hence the work of deformation is less by this arrangement than it would be by supposing less rigid parts to carry it. This is the principle of least work, which states that of all the various ways in which a given elastic structure could be supposed to be deformed in carrying a load, the way in which it will actually be deformed is that in which the load will perform the least work in deforming the structure.

Fig. 83, showing, as it does, merely a ground plan of the lines of force cannot represent fully the distribution of the points of attachment of the lines of force which are generated all over the surfaces of the rods and start out everywhere like the quills of a hedgehog, in greatest number at points where the two rods are nearest to each other, until in the immediate neighborhood of the point of contact between the rods, the concrete grips the rods and holds them with a firmness and strength that would seem incredible did not experience demonstrate its safety and reliability.

5. Mechanics of Embedment. The mechanics of reinforced concrete under flexure may be summarized as follows:

The co-operation or combined action of the two materials, concrete and steel, to resist bending, depends solely on the bond between the two, which has been discussed briefly in a preceding article.

In the case of plain rods, this bond is in reality a shrinkage grip which prevents the steel from sliding thru the hardened matrix in which it is embedded, and the resistance afforded by this bond is subject to well defined laws which may be stated as follows:

The bond shear is zero wherever the tension in the steel is constant. It passes thru zero where the moment passes thru a maximum or minimum. It must be depended upon whether the reinforcement is in one direction only as in a beam, or in multiple directions as in the slab.

Bond shear generates stresses emanating from the surface of the bars which may be treated as lines of force. These lines of force follow the general laws of distribution of force thru any medium, that is, their intensity is inversely as the square of the distance from the surface of the steel on which they are generated.

These general laws enable us to investigate or follow out the part played by bond shear in the mechanics of a slab or beam. In the case of a simple beam in accordance with the law stated, the intensity of the bond shear for uniform load is zero at the center and greatest toward the end of the beam, and it is to the bond shear or the lines of force generated thereby that we may attribute the difference in the failure of an over and an under-reinforced beam.

In the case of the beam with light reinforcement, failure takes place at the center by the yielding of the steel. With heavier reinforcement, on the contrary, failure is more liable to occur toward the end by indirect tension induced by the bond shear which is greatest toward the end of the beam and which may be resisted only by the direct tensile strength of the concrete itself.

The deportment of the simple beam as affected by the stresses set up by the bond shear is of interest. In a newly cast beam, in the preliminary stages of the loading, the stress in the steel as determined by the extensometer is much less than that figured on the assumption that the steel only resists tension. In fact, it is only about half as great as we should compute the stress to be on the above basis, until the steel is stressed up to four or five thousand pounds per square inch. When the steel is stressed up to twelve or fifteen thousand pounds per square inch, the measured stress in the steel does not differ from the computed stress in the steel by more than ten or fifteen per cent.

With the slab reinforced in two directions, however, the phenomenon differs from that observed in the beam. Take for example the case of the slab reinforced in two directions, bent in such a manner that the rods in both directions are brought into tension at the same time. The indirect stresses generated by the two sets of rods will under this condition react upon each other. Since the lines of force diverge from each rod they may meet and coact thru the concrete as a medium of transmission of the stress, which is not possible in the beam with one way reinforcement, since in the beam these stresses cannot coact with each other in like manner and they are of one kind only and not two kinds acting in different planes as in slab action. This fundamental difference of the stress induced by the bond shear in a beam from that induced in a slab renders the two types of structure mechanically different and necessitates their treatment in a manner which takes into consideration the difference in the operation of the indirect stresses referred to.

Another difference lies in the presence of twisting resistance in conjunction with bending resistance which results in a mode of operation different from that in beams in that twisting is resisted by shearing stresses in the slab laterally, while bending is resisted by tensile and compressive stresses in the horizontal fibers of the slab acting as a couple about the neutral axis of the vertical section. The concrete without reinforcement is of ample rigidity to properly withstand twisting of the slab bodily about any horizontal axis as well as the twisting of the elementary prisms about vertical axes, while the direct tensile resistance of the concrete is inefficient as a dependable element in resisting flange stress.

It is thus seen that the concrete in the slab is made to perform a function in slab action which the manner of bending does not permit it to perform in beam action. Further, the shearing resistance in the concrete is a dependable element while the direct tensile resistance of the concrete or even the residual strains and induced stresses of shrinkage are not dependable for the reasons discussed in Chapter IV.

6. Distribution of Bond Stresses in Type IV Floors: The preceding discussion of the mechanics of the indirect stresses generated by bond shear indicates the general laws which govern the distribution of metal necessary to secure effective continuous plate action in type IV floors, *i. e.* flat slabs supported by columns.

At the diagonal center of a panel, the moment of the applied forces passes through a maximum and the bond shear accordingly passes through zero, hence an arrangement of narrow strips of reinforcement on diagonal lines from column to column is of no utility in securing lateral efficiency by the operation of bond stresses. Wide spreading reinforcement covering substantially the area between lines of inflection is the prime requisite for efficiency.

The force of this remark will be better appreciated from a detailed consideration of the moment curve of the circular suspended plate under uniform load.

The area of a floor of this type may be subdivided according to deformation into three divisions, see Fig. 29, Chapter III.

a. Cantilever area about the columns convex upward and approaching spherical curvature for equal column spacing.
b. Suspended circular plate concave upward about the diagonal center of the panel.
c. Saddle shaped areas between the cantilever areas and the suspended central plate.

Considering the suspended plate, the moment curve differs from the parabolic curve of a beam uniformly loaded in that the center of gravity of that part of the load transferred to the support is nearer to the support than in the case of the beam by one third. This renders the moment curve very flat at the center and the variation of the moment for a considerable distance each side of the diagonal center very small, and on the other hand it renders the moment curve correspondingly steeper toward the line of inflection, and hence effective lateral action of indirect stress must occur toward the the outer edge of the plate since the intensity of bond shear depends on the moment increment. The moment increment being nearly negligible for a considerable distance about the diagonal center and zero at the center, the efficiency of a narrow belt is negligible from the standpoint of plate action.

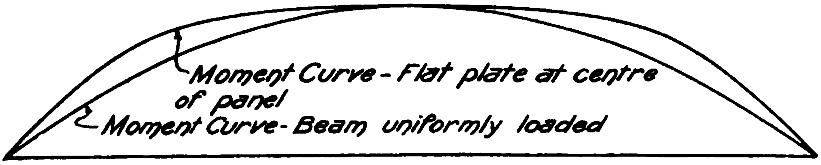

Moment Curve - Flat plate at centre of panel
Moment Curve - Beam uniformly loaded

Fig. 84.

Thus the arrangement of the reinforcing steel laterally as well as vertically vitally affects the characteristics of the structure in such wise that when the belts of rods which run in multiple directions from support to support are spread out sufficiently to make the metal cover the entire area with crossed reinforcement permitting coaction of bond stresses it imparts the property of plate action to the slab in which circumferential resistance occurs in wide areas about the columns where the metal is at the top and also in the central area of each panel where the metal is at the bottom.

Without such wide spreading belts full and effective plate action is impossible. Narrow belts, or belts part of them in the bottom of the slab at the columns, can not produce the success of the wide belts of the Mushroom structure.

Because of the greater steepness of the moment curve as the line of inflection is approached the action of bond shear in producing the imitation of plate action occurs in the rim of what might be called the dish shaped areas shown in those portions of Fig. 85 which are shaded red. The dotted lines indicate the line of weakest section in the original four-way Mushroom construction. Much improvement has been made in the later type of flat slab floor, termed the Spiral Mushroom, which will be described later.

FIGURE 85

ILLUSTRATION SHOWING IN RED THE REGIONS WHERE IMITATION OF PLATE ACTION OPERATES EFFICIENTLY.

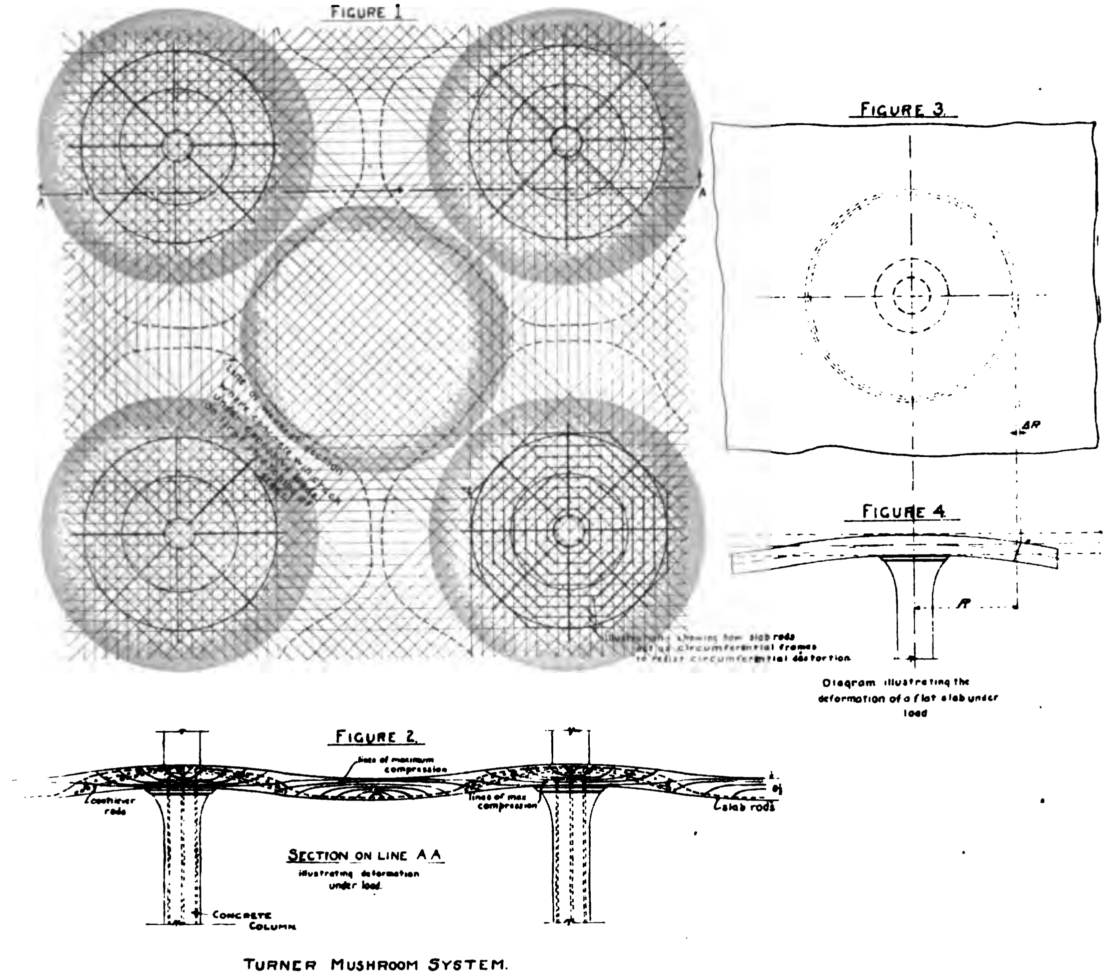


FIGURE 1
A
A
Illustrations showing how slab rods act as circumferential frames to resist circumferential distortion
FIGURE 3
ΔR
FIGURE 4
R
Diagram illustrating the deformation of a flat slab under load
FIGURE 2.
lines of maximum compression
cantilever rods
lines of max compression
slab rods
SECTION ON LINE A A
illustrating deformation under load
CONCRETE COLUMN
TURNER MUSHROOM SYSTEM.

The colored portions over the columns show the convex areas in which cantilever action occurs. The red circle about the panel center covers the dish-shaped area concave upward around which the plate action occurs. The circles over the columns represent the cantilever portion and the outside of the circles represent the limit of the cantilever span. The outside edge of the circle about the panel center indicates the limit of the suspended span which is held up by the cantilevers and the saddle shaped areas directly between columns. Because bond shear is greatest at the end of a suspended or supported span and at the end of the cantilever, imitation of plate action occurs efficiently only at the outside or rim of these inverted and upright dish-shaped areas. But imitation of plate action in a concrete slab differs somewhat from the mode of operation of homogeneous plate in that the distribution of bond stresses is such that the center of action of the circular stresses has a larger radius than in a homogeneous plate thus increasing their efficiency. This advantage can be secured only by a wide mat of crossed rods covering the lines of inflection. Narrow belts cannot produce this kind of resistance.

7. Resistance to Deformation Furnished by the Bond Stresses of Crossed Rods in Slabs. Where two series of uniformly spaced crossed rods are embedded in a concrete slab of uniform thickness bent in dish shaped curvature so that both series of rods are under nearly equal stresses they coact with each other through the matrix horizontally in the manner illustrated in Fig. 83, Section 3. In such lateral coaction the tensions and compressions produced in the concrete by the horizontal shearing forces along any rod are divided between those that are held in equilibrium laterally by similar indirect stresses generated by bond shear along horizontal rods normal to the rod under consideration and those that are held in equilibrium by the horizontal compressions in the compression zone above the tension zone which the rod strengthens.

Those portions of the indirect stresses due to bond shear which act between the rods alone and are held in equilibrium laterally constitute a series of localized self-contained systems. Linked together they constitute an essential part of the resistance to the applied circumferential stress. Their action is that of connecting the rods in such wise as to enable them to do their part in resisting circumferential tensions, and such circumferential tensions in simple synclastic curvature operate in a manner independent of the deflections and hence are independent of the propagation of radial stresses from the center of the panel outward in the suspended span or toward the central support in the cantilever portion of the slab. Those portions, therefore, of the horizontal shears that coact with each other laterally in the tension zone are independent of the compression zone radially and consequently play no part in propagating the tensions along the length of the bars radially in the manner of beam action. The separation of this portion of the shear from that which propagates such tensions will now be investigated.

The intensity of force emanating from a point in an elastic body at a given distance from that point, varies inversely* as the square of the distance from the point whence the force emanates. When the force emanates from a series of points on a line the intensity of the resultant force upon a plane at different perpendicular distances from the line is inversely as the distance from the line because of the crossing or superposition of the lines of force emanating from the several points along the length of the line.

A bar corresponds measurably to a line and although it has a certain size diametrally the deviation from the law of force emanating from a line will not be material for the purpose of our comparison of the distribution of bond stress as between that portion held in equilibrium laterally and that portion which is held in equilibrium by coaction with the compression zone.

Laterally the stresses emanating from corresponding points in two intersecting rods coact with each other and meet on a plane bisecting the angle between the rods. Because the length of the path is twice the distance to the bisecting plane the relative rigidity under the action of these stresses is directly as the distance apart of the bars so long as E_c remains constant and the material is not stressed beyond the limits of elasticity.

The relative rigidities of the paths determines the relative amount of those stresses that coact between intersecting rods, as compared with the amount that coacts with the compression concrete above. The distribution of the latter is shown in Fig. 82 in which tensions and compressions act at 45° to the rod and are of practically uniform intensity for such a short distance along the bar as the length of a mesh. This distribution is quite different from the variable distribution in a horizontal direction and the separation in the horizontal direction of the tensions on one side of the intersection or crossing of the bars and the compressions on the other.

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*This is also the law of distribution of the force of gravity and of the intensity of light, etc.

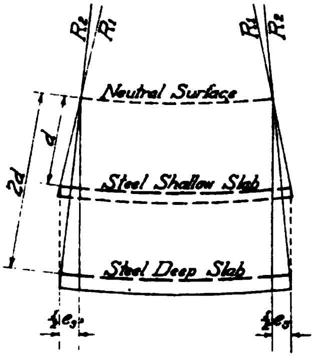

Fig. 86

The rigidity of the path by which the indirect stresses travel in a vertical direction must be investigated in order to compare the relative rigidity of the horizontal and vertical paths of distribution. It will be noted in Fig. 86 in a beam in which the depth of the steel is d below the neutral plane that for a given elongation e_s in the steel the horizontal shearing deformation due to the stretching of the tension zone is twice as great per unit of depth as would occur in a beam in which the distance from the neutral plane to the steel is $2d$, and the elongation of the steel in tension is also e_s. Therefore, the rigidity of the vertical path traversed by the indirect stresses which produce these shear deformations is inversely proportional to the distance from the steel to the neutral plane and under the principle of rigidities the distribution of indirect stress horizontally and vertically will be proportional to the relative rigidities of the respective horizontal and vertical paths of distribution.

From the foregoing principles the distribution of bond stress vertically and laterally may be apportioned inversely as the distance from the steel to the neutral plane as compared with half the diagonal of the rectangular spacing of the bars. This may be taken to hold true in case the spacing of the bars is equal to, or greater than 1.4 times the distance from the steel to the neutral plane.

We may now estimate as nearly as we can the relative distribution of bond stress laterally and vertically upon this basis in the case of a slab in which the distance from the center of the steel to the top of the slab is 8″ and the bars are 7/16″ round, spaced 7″ at right angles to each other in each direction. In such a slab by referring to the j and k curves of Chapter VI, it will be seen that the neutral axis would be 5″ above the steel. The distance diagonally across the mesh between the mid points of adjacent sides of this 7″ mesh is also practically 5″ so that the rigidities of the vertical and horizontal paths may be taken to be equal. Therefore, the distribution under this law would be one half in the lateral direction and one half in the vertical direction.

We will compare this distribution with the result which would be brought about by disposing the same weight of steel in the form of ⅝" rods instead of 7/16" and spacing the rods twice as far apart. The increase in the diameter of the bars reduces the distance from the center of the steel to the neutral plane to approximately 4 and ⅝ inches and since the bars are twice as far apart the ratio is nearly as 1 to2. Therefore 2 times as great an amount of bond stress operates in the vertical direction as in the lateral and the relative stress per square inch on the bars would be as ⅔ to ½ or approximately 1⅓ times as great.

The assumption upon which this relative stress is computed is based upon a radial section bisecting the angle between the two series of crossed rods. At other points around the circumference dependent on the angular direction of the crossed rods relative to the supports the distribution of bond stress in a vertical direction will be greater and that in the lateral direction less and *vice versa* with further angular change making an average approximately that which we have computed above. Thus the mat of crossed rods wherein the rods are normal to the sides of the panel are equivalent in their resistance to radial and circumferential stresses to that of a mat in which the directions of the crossed rods are parallel to the diagonals of the panels.

It may be observed that bond stresses in a mesh length are distributed with approximate uniformity along the length of the bar in their coaction with the compressions above the neutral plane vertically with lines of indirect compression and tension crossing each other at 45 degrees in that region, whereas, in a horizontal direction indirect tensions and compressions are separated from each other, operating on opposite sides of the bar in such wise that as the compressions decrease on one side of the bar, the indirect tensions increase on the other side, so that the resultant effect is a relative uniformity of distribution of all the bond stresses coacting laterally in a mesh length very like that found in the vertical direction.

It might be thought because of the increase in the intensity of the tensions toward the apex of the mesh in one direction on one side of the bar and of the compressions in the other direction on the other side of the bar that the elasticity of the concrete might be exceeded at and near the apex or points of contact of the bars. At this point, however, the bars are not opposite each other but are disposed vertically one above the other. Consequently, the bond stresses at and in the immediate vicinity of a point of contact,

act predominantly in a vertical rather than in a lateral direction, so that their lateral intensity is limited to an amount within the elasticity of the concrete when subjected to reasonable test loads.

We will now compare the case of the lateral and vertical coaction of the rods where the spacing is decreased from 7″ to half that amount, or to 3½″ center to center in each direction, the rods still remaining of the same total cross sectional area. With this spacing the vertical distance from the steel to the neutral plane would remain about as before, 5″, while from rod to rod the total diagonal of the mesh would be nearly 5″. Or the half diagonal is half as great as the vertical distance from the steel to the neutral plane, as shown in Fig. 87. Because of this close spacing of the rods there is the super-position of indirect stresses vertically shown in Fig. 87 while no such super-position can occur laterally. Therefore, when the spacing is made half as great there would be a super-position of the bond stresses vertically upon the neutral plane and this path would then have become twice as efficient. Hence in light work, there is little to be gained by spacing the reinforcing elements closer than the distance from the neutral plane to the steel. Where, however, the slab is heavy, it is better not to exceed six or eight inch spacing since in the heavy slab larger rods are required and greater intensity of bond stress is developed and good design requires that the intensity of the bond stresses be kept to a minimum by more uniform distribution thereof.

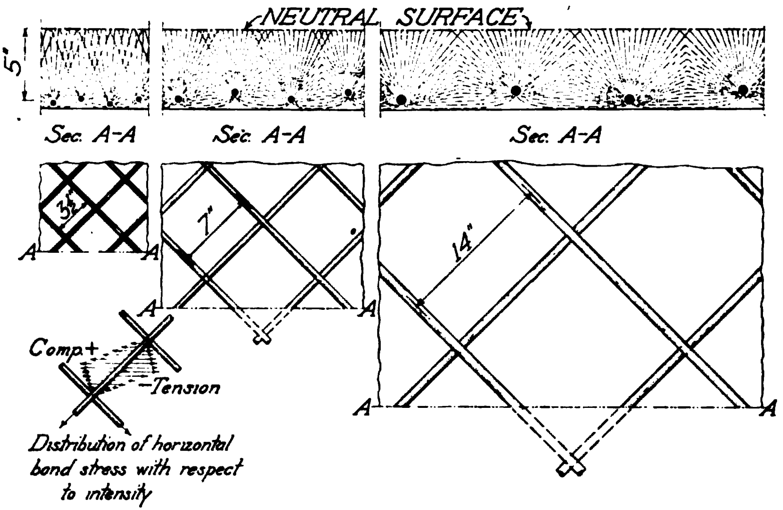

Fig. 87

8. **Analogy Between a Composite Plate of Concrete and Metal and a Homogeneous Plate:** In a column supported homogeneous plate, such for example as would be presented by a boiler plate supported by stay bolts, there would be the same dish shaped curvatures that were discussed in Chapter III. The resistance to deformation will take place by circular and radial stresses about the diagonal center of the panel and about the supports. The intensity of the radial stresses would be greatest at the diagonal center of the panel in that portion of the plate that is concave upward and the circumferential stresses also would be greatest adjacent to the center. The conditions in the composite plate differ from those in a homogeneous plate because circumferential resistance is localized in the rim of the dish shaped area in the composite plate since it depends on bond shear which is present in these areas and absent at and about the center. The locus of the centers of action or centroid of the circumferential stresses thus has a larger radius than that of similar stresses in the homogeneous plate, while the reinforcing elements near the middle of the dish shaped areas are thus strained much more uniformly than the fibers of a plate of homogeneous material under similar loading.

Because the resultants of the circumferential tensions and compressions act at larger radii in the composite plate of concrete and steel than in a plate of homogeneous material they also act through a greater circumference than they do in a plate of homogeneous material, and consequently they are correspondingly more efficient in reducing the maximum intensity of the stresses at mid span as measured by the extensometer. This increase in efficiency is an increase comparable to but not identical with Poisson action in homogeneous material.

In the composite plate, however, because of the low value of K for the concrete there is very little true Poisson action. If K were .2 it would amount to a reduction of only four percent of the compressions in the compression zone. The fact that the embedment of reinforcement increases the coefficient F of shearing resistance tends to reduce even this small amount of true Poisson action. On the other hand on account of the concentration of the circumferential resistance in the rim of the dish shaped area, the action of bond stress causes an increase in efficiency equal to that which would be produced by a high value of the coefficient K in a homogeneous material. For fine grained reinforcement with a spacing not exceeding the distance from the center of the steel to the neutral plane Eddy* found that this action was equivalent in its effect to a coefficient $K=.5$ in a plate of some imaginary homogeneous material when it was bent to the same curvature as the composite plate of concrete and steel was bent, thus producing a reduction amounting to 25 percent in the steel stresses by figuring the reinforcement as a sheet in the area of synclastic curvature, where the stress would be reduced in the ratio of 1 to $(1—K^2)$.

A good analogy of the operation of a fine grained mat in concrete in its imitation of the gradation of intensity of stress from point to point in the composite plate is found by comparing its effect to that of the screen used in making a cut of a photograph for use in printing, to imitate the gradations of light and of shade in the original photograph. When the screen is fine the eye can detect no difference in the reproduction on glazed paper from the original photograph. When a coarser screen is used for printing on rough paper as in newspaper work the gradations of light and shade are less perfectly reproduced while if any of these cuts be examined under the microscope nothing but a series of disconnected dots appears with no likeness whatever to the photograph such as appears when viewed in their natural size. and relation. Just as no reproduction of the photograph can be produced with too coarse a screen so no reproduction of the gradation of stress in plate action can be imitated by too coarse a mat having too wide a spacing between the rods.

9. Substantiation of the Foregoing Principles By Test Data: The extensometer test of the Shredded Wheat Building at Buffalo, N. Y., by W. S. Slater, described in the Journal of the Concrete Institute, October-November 1914, furnishes verification of the foregoing analysis in that the moment coefficient of the steel stress in the respective directions under uniform load was twice as great as the corresponding stresses in the tests of the Franks, Larkin, and Deere & Webber Building referred to in Chapter II and as tabulated page 1561 Transactions American Society Civil Engineers, LXXXII.

Nine panels of the Shredded Wheat Building 20×22 feet were loaded. The slab was 7″ thick in the central portion of the panel and increased to a thickness of 9″ in an area 8′6″ square around each column. The columns were octagonal 25″ in diameter. The reinforcement of the panel in the central portion consisted of ½″ round corrugated bars at 11½″ centers the long way of the panel, and $9\frac{1}{4}''$ centers the short way, bent up abruptly at about nine feet from the center line between columns to the top of the slab as shown in Fig. 88.

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*Flat Plate Theory of Reinforced Concrete Floor Slabs, Eddy 1913.

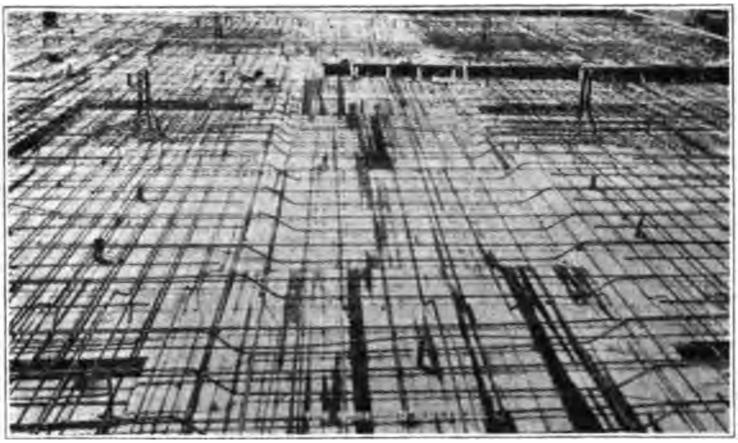

Fig. 88

Referring to the j and k curves in the preceding chapter, the center of the steel would be 3.6″ from the neutral plane. The diagonal of the reinforcing mesh was 14.5″ and the half diagonal twice the distance from the center of the steel to the neutral plane. Therefore we have the same relation as that with the $\frac{5}{8}''$ round rods compared with the 7/16″ rods in the previous example and the same relative division between lateral and vertical bond stress.

From the location of the bars in the slab it will be observed that only seven of these bars are in contact with those at right angles with them in the central portion of the slab and that accordingly whatever circumferential action takes place must be nearer to the diagonal center of the panel than would be the case were these rods carried to a greater radial distance from the center of the panel, and because of this arrangement the increased efficiency brought about by moving the centroid of circumferential action to a greater radius is lost, so that the efficiency of the steel would be reduced not only by the wide spacing but also by the loss of this additional element of strength to an amount expressed by the ratio of 1 to $1-K^2$ found for such slabs with a fine grained reinforcement in mats covering the areas of synclastic curvature.

A computation of the relative lateral and vertical rigidities in the floor of the Shredded Wheat Building, shows that $23\frac{1}{2}$ percent of the bond stress would be distributed laterally and 76½ percent vertically as against 62½ percent laterally and 37½ percent vertically in a slab of the same thickness and cross section of steel distributed scientifically in fine-grained reinforcement.

This computation is fully corroborated by an analysis and numerical comparison of the results of published tests, which show that the metal in this floor has only about one half the efficiency disclosed in many other published tests.

10. A Given Bending Moment will Bend a Flat Plate Synclastically less than Half as much as it will Cylindrically. It has already been shown that in a circular plate, or in a piece of a continuous plate, so situated around a point of symmetry that it acts like a circular plate, the radial and circumferential stresses called into play at any point by symmetrical loading are equal to each other. But where this plate has a dish shaped curvature the magnitude of these resisting stresses compared with the applied forces is a question about which a detailed investigation is given later in this treatise. It will be sufficient for our present purposes to picture to ourselves a circular piece of a plate which has an approximately spherical curvature at its center point O. Pass a plane XY tangent to the surface at O and let OZ be normal to it at O. Then if a bending moment be applied about OX it will not only make the curvature of the plate greater in the plane ZOY but at the same time it will make its curvature less in the plane ZOX to an equal amount as is evident by experiment on a shell of any elastic material, and as is proven in Gauss' theorem of the curvature of thin shells. Since the force applied to produce the given bending moment must produce both these equal changes of curvature simultaneously by producing elongations and compressions in twice as much material as in a plane plate of equal cross section, each of them is only half as great as would be produced in a plane plate in a single direction by this same moment. Hence it appears that the extensometer deformations produced by an applied moment are not more than half as great in a spherical dish shaped plate as in a plane plate, because in the latter the elongations are in one direction only, while in the former they are in two directions at once, at right angles to each other. In a homogeneous convex plate the actual deformations are less than one half of those in cylindrical curvature by reason of a further reduction due to Poisson action, and in a reinforced plate the stresses are made less than 50% of those due to cylindrical curvature by the transfer of the centroid of the circumferential stresses by the bond shear action to a greater distance from the center than in a homogeneous plate. So that with a fine grained mat, the stress due to the applied moments is not more than half as great in a radial direction and not more than half as great in a circumferential direction as in case of cylinderical curvature, but because of the operation of the circular stresses on the whole at a greater radius than would be the case in a plate of homogeneous material, in which $K < \frac{1}{2}$, the maximum radial and circular stresses are both still further reduced in magnitude by 20 to 25 percent.

Taking the smaller figure to cover a somewhat wider range of variation than would be covered by a reduction of 25 per cent, we would have the stresses in a relatively fine grained mat amounting to not more than 0.4 as much in the steel in one direction as would be required to resist the applied moment when treating the applied moment in the manner of beam theory.

11. The Equivalence of a Diagonal and a Rectangular Mat of Rods. In the case of a rectangular arrangement of crossed rods, one set only would cross any section parallel to column lines at mid span and we need to provided on this basis only 0.4 of the metal in the dish shaped area that would be necessary on beam theory to resist the applied moment there found, while in the area of saddle shaped curvature, with parallel rods, the full cross section of steel would be required according to beam theory to resist the moment there applied. Where the mat about the center of the panel is formed by two series of diagonal crossed rods of proper width a similar section would cut all of these rods at an angle of 45° and the efficiency of the metal normal to the section would be 1.414 times the cross sectional area of one belt, but whereas these rods have in the direction of their length to resist only 0.4 of the moment applied along their length, at an angle of 45° they would have to resist a proportion of that moment of $2 \times .4 \times \cos 45°$ or $.8 \times 707 = .566$ of the moment. We see that the ratio of 566 to 1 414 the relative cross sectional area equals the ratio of 0.4 to 1, or for the dish shaped area of synclastic curvature we may consider the metal $2\frac{1}{2}$ times as effective as a single layer of rods normal thereto, whether the mat be made up of rods crossing each other in a rectangular direction or crossing each other in a diagonal direction. Equivalence under uniform load does not establish equivalence under non-uniform load as will be shown later.

12. Early Development and Diverse Present-Day Views of Flat Slab Analysis: The fact that a reinforcing rod can itself resist tension along its length only, early led to the firm conviction that crossed rods in a concrete slab could not possibly produce any greater effect than the aggregate of all the rods taken separately. That this was the consensus of opinion among the engineering profession fully appears from the discussion by twenty members, of the paper of Captain John S. Sewell, Trans. Am. Soc. C. E. Vol. LVI, 1906, in which he made the statement that no economical system of concrete flooring could be constructed without rolled steel beams or concrete tee-beams. Turner only, of twenty members discussing this paper took issue with Capt. Sewell on this point, and disclosed a design embodying the principle of the mat which was at that time being incorporated in the first commercially successful flat slab and column construction.

The difficulty of properly analyzing this type of construction forced Turner to adopt at first the simple method of proportion discussed in Chapter II, in order to arrive at a successful reduction of his new construction to practice. This method was first published by Turner in 1909. This was followed in 1913 by Eddy's adaptation of the classical flat plate theory to the exact analysis of the problem of the flat slab uniformly loaded on rigid columns. This analysis was based upon the consideration of the deformation of an imaginary homogeneous plate which would deflect or bend in the same dish shaped curvature as a non-homogeneous slab. Given the elastic coefficients of this imaginary material the geometry of deformation could be determined mathematically from the flat plate theory and the linear deformation or change in shape thus found was then translated into stress from the known elastic coefficients of concrete and metal for the composite plate. As a mathematical invention to arrive at the desired result, it seemed to the junior author very ingenious. It gave rise, however, to much real or pretended misunderstanding, perhaps because it was not carefully studied. Notwithstanding the fact that there have been numerous extensometer tests of flat slab and column construction and that it has, since the first disclosure by Turner, of the principle of the mat in imitating flat plate action, been widely introduced, the conflicting opinions regarding its proper analysis are almost legion. Some of them are curiously inconsistent. Thus the steel stress moment of the Shredded Wheat Building which is one hundred percent greater than that found in the Franks, Schulze and Larkin Buildings, is accounted for by the Joint Committee member Hatt, on the ground of the tensile strength of the concrete, common to all. The Shredded Wheat job had the stronger concrete, the initial modulus found being 5×10^6 (on short time test), but figured at only 3×10^6 (Slater, Am. Conc. Inst. 1914). But the Shredded Wheat job gave the highest steel stress instead of the lowest.

Mr. Hatt, Final Report Joint Committee Trans. Am. Soc. C. E., Vol. LXXXII, p. 1560, says:

> "It is evident that, even at steel stresses when the concrete is broken down in tension at some sections there is a marked discrepancy between internal and external moment." (Amounting to 350 to 500%).

Quoting further, page 1562:—

> "The evidence at hand points, probably, to a more 'persistent operation' of the tensile stresses in the case of flat slabs with about half of one percent reinforcement than in beam structures which are more heavily reinforced......
>
> "Individuals will no doubt attribute these remarkable discrepancies between steel and internal moments, appearing in steel strains and external moments, to some other action than the tension in the concrete. The writer would rather look to the latter than to some poorly defined and unproved agency."

Thus extensometer measurements are universally conceded to demonstrate the existence of a state of stress in column supported flat slab floors radically different from that found in beam action.

To conclude from this evidence that there is a "more persistent" operation of the tensile resistance of the concrete in the case of flat slabs than in beams after the concrete has admittedly, become broken down in tension and the particles separated, is as irrational as to expect a broken chain or cable to "persistently" resist after parting, the tension that caused it to part.

In developing a general theory embodying the principles which are applicable broadly to a wide divergence of details, considerable agreement should be found with specific analyses developed to cover specific details. For example, consider the steel arrangement recommended in Taylor and Thompson's work on Plain and Reinforced Concrete, page 484, 1911 Ed.

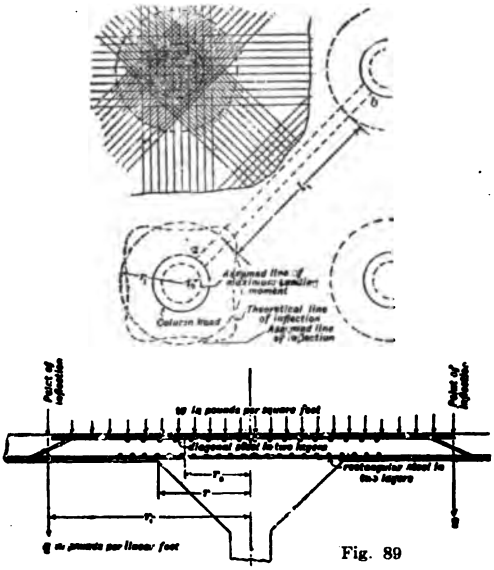
r_1
r_0
Column head
Assumed line of maximum tangential moment
Theoretical line of inflection
Assumed line of inflection
Point of inflection
Point of inflection
w in pounds per square foot
diagonal steel in two layers
rectangular steel in two layers
r_0
r
r_1
q in pounds per linear foot

Fig. 89

From the note on the cross section of the reinforcement it will be observed that the diagonal bands only are carried up to the top of the slab and over the columns while the direct bands are kept in the bottom throughout. Thus over the columns the mat of crossed rods is a small square not much larger than the outer diameter of the capital. Its area does not extend from the edge of the capital more than about one sixth of the distance in a diagonal direction to the line of inflection, as shown in the lower part of the figure. Consequently, the mat over the columns could furnish only negligible circumferential resistance. The mat about the diagonal center of the panel between the columns is likewise so restricted in area that the advantages of the operation of a mat are greatly reduced, with the result that two to three times the cross section of the steel will be required that would be necessary were the areas of crossed rods of proper length and breath We find that Messrs. Taylor & Thomson's analysis of the construction of this specific type requires, as does the author's analysis, between two and three times the cross section of steel necessary for the distribution adopted by Turner in the original Mushroom construction.

Let us now investigate the specific details illustrated in Fig. 90.

In this flat slab type the rods are large in cross section spaced relatively far apart and the points where they dip downward are staggered thus preventing the mat from extending in a broad area about the center of the panel. With this coarse grained arrangement of reinforcement nearly full advantage of the twisting resistance would be secured which would not be the case in the arrangement in the specific details recommended by Taylor and Thompson. On the other hand, the coarse grained distribution of the metal and the relatively small diameter of the mat of crossed rods about the center of the panel would necessitate its operation more largely in a radial than in a circumferential direction under the principles before investigated and nearly twice the weight of steel would be here required that has been found necessary where a fine grained distribution of the material has been adopted. For these specific details the designers have very wisely used a large excess of steel over that essential in the pioneer flat slab construction. These designers, from experience in their customary use of such distribution of metal as above shown, have erroneously attempted to draw the general conclusion that the use of such an excess is required regardless of its manner of distribution.

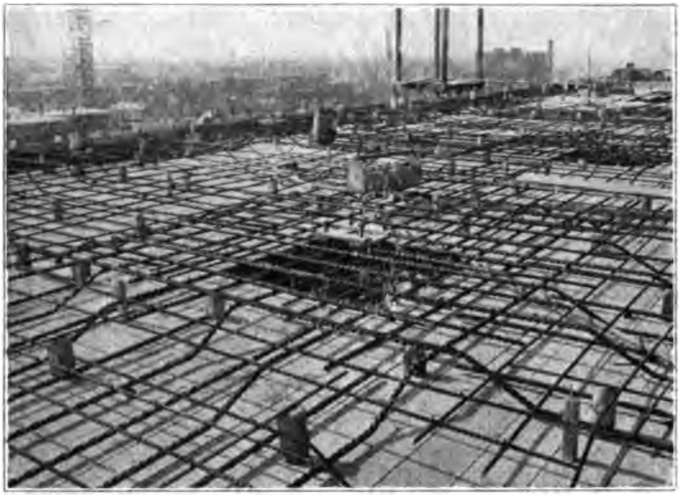

Fig. 90

13. Bond With Deformed Bars. Mörsch discusses the action of deformed bar reinforcement thus:*

> "In America various forms of reinforcement are employed all of which are designed to prevent slipping of the rod in the concrete. In the Ransome rod, this is secured by twisting the square steel bar; in the Johnson bar, elevations on the surface of the rods are produced in the rolling; and the Thacher or knotted bar is provided with swellings, while maintaining a constant sec-

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* Concrete Steel Construction p. 17.

> tional area. These 'knots' may well have the desired effect when the rod is anchored in a large mass of concrete, but they will act in an opposite manner in the small stems of T-beams, especially at their bottoms, where they will have a splitting effect and thus cause premature failure of bond. It will be shown later that the adhesion in the case of ordinary round rods with hooked ends is ample to transfer all actual stresses, and furthermore, the arrangement of the principal reinforcement may be so designed with respect to the shearing stresses that no occasion should arise to make up any deficiency through the use of those costly special bars."

As regards the mechanics of embedment, the effect of the "knots" as Mörsch terms them, would cause slight irregularities in the intensity of the indirect stresses, assuming that there was any over-strain tending to cause the bar to slip and to bring into action the mechanical bond. This variation from the condition of uniformity of bond shear with the plain bars would not be material under loads producing stresses not exceeding those which are safe, tho the disadvantage suggested by Mörsch in the case of the thin ribs might be looked for under loads approaching the ultimate strength.

It has been found that a piece of polished cold rolled shafting has about one half the bond that a rolled rod with black mill surface has. Therefore, it appears that a difference in roughness makes a great difference in the bond resistance and leads to the conclusion that the best form of deformation is that of relatively small projection rather than of large widely spaced deformations. From the tests quoted in Article 9, Chapter 1, it may be noted that the round bars gave greater adhesive resistance than square bars. Because the concrete shrinks around the bar in hardening, there is evidently a localization of shrinkage stress on the square bar, rendering that form of cross section less satisfactory than the round bar while flat bars in the tests referred to give very low values indeed. Longitudinal ribs act in the same manner as corners, localizing the shrinkage stresses and rendering that form of section objectionable and less satisfactory than the round. Wedge shaped changes in the section of the bar are objectionable for the reasons pointed out by Mörsch. Viewed as a tensile reinforcing element, large deformations or irregular changes in the form of a bar are a source of weakness both from the standpoint of possible damage in bending the bar and because planes of weakness are frequently produced in bars of irregular shape during the process of rolling. While deformed reinforcement has little value in buildings where the concrete sets in air, it is desirable in subaqueous work. There are, however, many who favor deformed reinforcement even under the former conditions and the foregoing discussion may serve as a guide to the selection of the most satisfactory type of section.

14. Summary. The mechanical action of the concrete matrix as a field of force which by the action of bond stresses between it and the steel reinforcement brings the crossed rods into mutual action as a mat is here developed in some detail. And in particular its action in a reinforced plate differs from that of the stresses in a homogeneous plate in virtue of the fact that their effective action is principally in the rims of the dish shaped areas where they cooperate to resist the applied circumferential tensions.

Synclastically deformed areas of flat plates when subjected to single applied bending moments suffer less than one half as much additional deformation as would be produced in flat plates by the same moments according to beam theory. This diminution of deformation in conjunction with localizing the circumferential tensions near the rim of the synclastic areas reduces the steel stresses in such areas to less than 0.4 of those required by beam theory to resist by direct tension the actual applied bending moment which in turn is less than indicated by beam theory because of the twisting resistance of the plate when bent in double curvature. Since rim action is more effective than Poisson action, steel stresses in general are reduced more below beam theory, in well reinforced plates, than in homogeneous plates.

The principle from which the remarkable strength of the flat slab arises is that of a mat of crossed rods covering the full area thereby furnishing the advantages of radial and circular resistance to oppose the applied moments of the load. A part of the remarkable strength exhibited is brought about by the presence of internal twisting resistance which hold part of the applied moments in equilibrium. The development of the full possibilities of this combination requires arrangement of the metal in mats covering the two tension zones, the one in the top of the floor over the supports and the other in the bottom about the panel center, for without such continuous top and bottom arrangement of metal synclastic curvature over the tops of the supports and about the central portion of the panel is impossible.

The effectiveness of this resistance depends also on the fine grained nature of the reinforcement, that is, upon its distribution in the form of small rods closely spaced. Wide spacing or unreinforced areas of any considerable magnitude cut off the action of indirect stresses and largely destroy slab action. The manner of support and the distribution of rigidities must be such that synclastic curvature is secured about the diagonal center and over the supports because this phenomenal resistance cannot be developed under cylindrical or anticlastic curvature.

CHAPTER IX.

DEFLECTIONS AND MODE OF OPERATION AS SHOWN BY CONTOUR LINES. DISTRIBUTION OF REACTIONS AND MOMENTS AS SHOWN BY LINES OF ZERO SHEAR.

1. Because the external work of the load is equal to the internal work of deformation under loads within the limits of elasticity of a floor, deflection is a function of the internal strains and has certain definite mathematical and mechanical relations to them.

In a plate bent under load in synclastic curvature in which the upper surface was a true horizontal plane before bending, we might find the position of lines connecting points having the same elevation by pouring water into the depressions formed under deflection and plotting the position of the water marks. But to experiment in this way would require a true horizontal surface which would be difficult to secure. For purposes of investigation, all that is needed is to find what change from the original vertical elevation the various points in the member have undergone. After loading and determining this change in elevation at a sufficient number of points and plotting the same, if contour lines connecting all points which have undergone the same vertical change in position are drawn at elevations which differ by equal vertical amounts the map of these lines correctly represents the change in curvature of the member produced by the applied load. The change in elevation produced by the load at any point may be measured with great precision by the use of vernier scales on sliding vertical rods or by gage dials properly connected to such rods. Since the upper surface of a floor is occupied by the load, test measurements of deflection may be made most readily from below.

When members under transverse stress are restrained by the manner of the support in different ways the different characteristic modes of operation resulting therefrom are clearly disclosed by the relative position and curvature of the contours under uniform load and the direction of the lines of zero shear, among which we may enumerate the following:

I. When a plate is bent in cylindrical curvature between parallel level supports, it operates on the beam principle. The contours are then a series of straight lines parallel to the supports, and all the lines of zero shear would be parallel, and normal to the supports, except one parallel to the supports at which the moment is a maximum. See Fig. 100.

II In a circular reinforced concrete plate supported around its outer edge and having a uniform moment of resistance in all directions comparable to a homogeneous plate, the contour lines would be a series of concentric circles about the center and the lines of zero shear would be a series of radial lines from the center to the supporting walls. See. Fig. 95.

III. A floor slab of Type III supported on four sides by beams dishes downward. The contours differ from circles in that they are flatter toward the sides and curve more sharply across the diagonals of the panel. In case of square panels the lines of zero shear radiate from the center. See Fig. 91.

IV. In floors of Type IV the convex surface about the column and the concave surface about the diagonal center of the panel are indicated by approximately circular contours and the typical saddle shaped areas along direct column lines is indicated by the 45° contour lines in the Figure representing Type IV at the right. The lines of zero shear radiate in each quarter panel from the column center. See Fig. 92.

Where contour lines are straight and parallel to each other, horizontal stresses of bending normal to these lines occur with no stress whatever along their length. Wherever there are curved contour lines about a given center convex outward the material is compressed

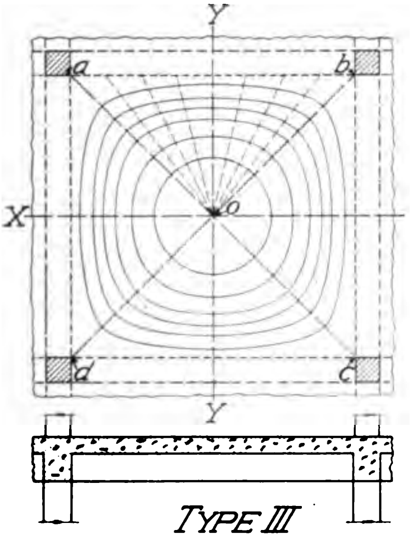

Fig. 91

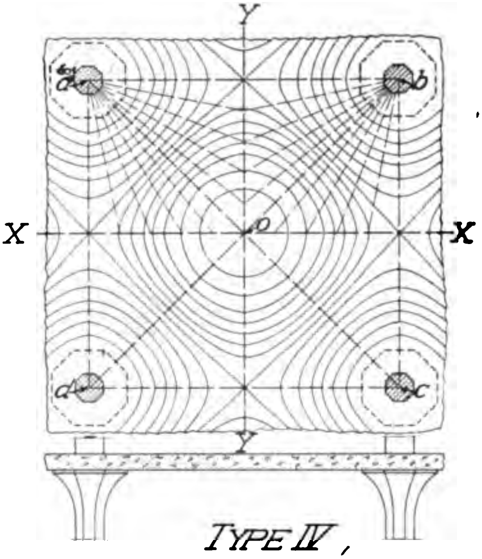

Fig. 92

radially and circularly on the compression side of the plate and is stretched radially and circularly on the tension side of the plate about the center. Points at which contour lines draw nearest to each other are points of greatest slope, and indicate points of inflection in the vertical sections of the slab.

In Case of Symmetry about the median lines of a panel a curved contour indicates there are stresses both normal to the median line and coincident with it along its length at the surface represented by the contour, and these stresses are principal stresses.

Thus the contour map becomes not only a graphical indication of the elastic deflection but shows at lines of symmetry the direction of the principal horizontal stresses. Observations determining deflection contours thus form an efficient method of experimental analysis in the investigation of distribution of stresses in concrete plates.

The contours previously illustrated are typical of plates in which there is a condition of approximate uniformity and symmetry of the reinforcing elements. Any localization of the rigidity of a floor by heavy reinforcement in strips changes the distribution of the stresses and the curvature of the contour lines in a manner whose general characteristics are well known and may be made clear by considering a circular plate uniformly supported at the edge and reinforced uniformly in the bottom throughout with small rods closely spaced at right angles to each other compared with another circular plate of the same dimensions and manner of support but reinforced with rods concentrated in narrow strips of great rigidity.

The contour lines of such a plate supported around the edge under a uniform load when uniformly reinforced would be a series of concentric circles about the center of the plate, further apart toward the center and closer together as the circular support is approached. The surface would be a surface of revolution approaching spherical curvature or more closely a parabolic surface of revolution with the apex at the center of the plate as in Fig. 95.

Consider now a plate of the same size, say 18′0″ diameter and of 6″ thickness, similarly supported and uniformly loaded, but reinforced differently, as follows:

Strips of six ¾″ rods spaced 2″ apart in strips at right angles to each other on diametral lines. A light reinforcement of 5/16″ rods 6″ on centers covering the rest of the area. The contours would no longer be circular lines concentric about the center. The curvature would no longer be approximately spherical but scallop shaped with ridges running along the lines of the concentration of reinforcement and hollows between them. The localization of the reinforcement in strips has changed the surface from synclastic to anticlastic across the lines of localized rigidity. See Figs. 93 and 94.

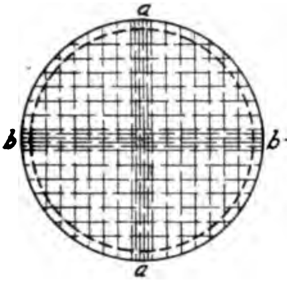

Fig. 93

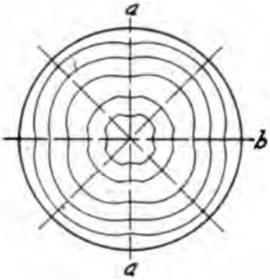

Fig. 94

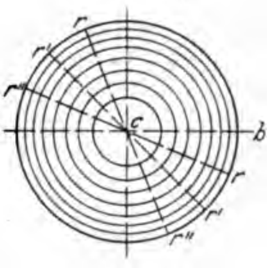

Fig. 95

2. Localization of Reinforcement Changes the Manner of the Storage of Internal Work. Where the reinforcement is localized in strips of relatively great ridigity, the contours of the surface which were circular in the evenly reinforced slab. Fig, 95, are bent inwardly making them convex toward the center across the strip instead of concave, as shown in Fig. 94. This produces the change from the characteristic contours of the circular plate with uniform reinforcement to that just described and causes an increase in the actual deflection of the plate as a whole because the storage of internal work is no longer divided between elastic resistance to radial and circular stresses.

Under load scallop shaped curvature occurs across the strips heavily reinforced as in Fig. 93. A state of synclastic stress such as is found in a uniformly reinforced slab with contours as shown in Fig. 95, is changed to an anticlastic state of stress with contours as shown in Fig. 94, which are alternately concave and convex toward the center of the plate. Now plate action depends on the storage of energy circumferentially by ring compressions on the upper and ring tensions on the lower surface of this plate, but when the dish shaped curvature of this plate is changed to a scallop curvature, tension and compression alternate circumferentially both above and below as the contours change from convex to concave toward the center. At the points where this change from tension to compression occurs there is no ring stress at all, consequently there can be no storage of energy in a circumferential direction to oppose deflections. Accordingly the mode of operation is reduced to that of radial resistance alone and the strength of the plate degenerates to that of mere beam strip action radially with correspondingly increased deflection and decreased ultimate strength.

Where the concentration of the reinforcement is less pronounced that is where there is but a small difference between the cross section of the metal in the strips and that of the average distribution between them such for instance as would be produced by the use of $\frac{1}{2}''$ rods in place of $\frac{3}{4}''$ in the strips and $7/16''$ rods distributed through the slab instead of $5/16''$, the change in contours would be less marked. Such moderate concentration would reduce the amount of ring compression normal to the strips of heavy reinforcement without developing the scallop shaped curvature sufficiently to make the circumferential stress change sign. The efficiency of circumferential action although impaired would not be entirely destroyed thereby until the sign of the resultant stress is changed.

3. Relation of Cross Section of Reinforcement That Resists Bending in Dish Shaped Curvature to the Form of the Contour. When the form of the contours indicates the presence of anticlastic curvature in circumferential directions which destroys the cooperation of circular and radial stresses it is important to determine the relative cross sections of the reinforcement which produces such anticlastic curvature.

If in a circular section described around the center of dish shaped curvature we plot to an appropriate scale the relative cross sections of the material upon different radii, by laying these points off consecutively in plan from the circumference of the circle toward the center and connect the points by curved lines, a figure is formed which is similar to the curve of the contour.

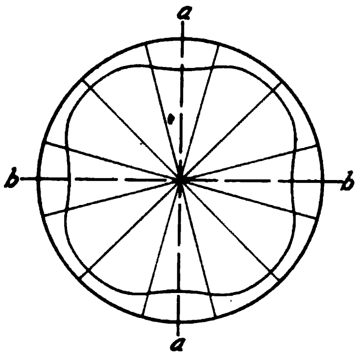

Fig. 96

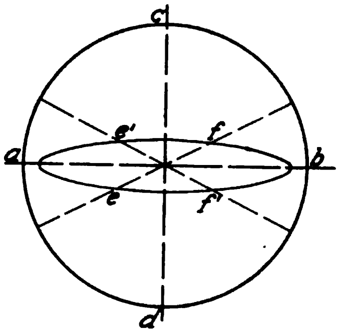

Fig. 97

Fig. 96 is a resistance diagram of the material corresponding to a non-uniform distribution of reinforcement in the circular plate supported at its edges as shown in Fig. 93. Fig. 97 represents the resistance diagram lengthwise and crosswise of a sheet of expanded metal. There is uniformity of strength in the diagonal direction *ef* and *e'f'*, but this uniformity in two directions does not control the resistance in all directions since any elongation along *e* toward *f'* would allow deflection along the axis *ab* substantially the same as though the resistance were that of beam action only. Consequently, a slab reinforced with expanded metal in one direction only cannot resist bending both radially and circularly but its operation is confined practically to that of beam resistance.

From the foregoing it appears that *plate action depends for its utility on substantial uniformity of resistance radially and circularly.* We cannot reduce the resistance in one direction greatly below that in another without changing the mode of operation in such wise that it degenerates into substantially that of mere beam resistance.

4. Warping and Twisting of Flat Plates. Let us consider the action of a single concentrated load W applied at the center O Fig. 98 of a thin square plate which is freely supported upon but not fastened to four rigid side beams that extend from column to column. Let the load be large enough to produce a center deflection of one or two percent of the span L. Under such deflection the corners of the plate would rise from their supports. The statical moment of the reactions at two supports like a and b on a single side about a median line XX of the plate would be $\frac{1}{4}WL$. Since the plate curves upward from the beams at and near its corners, the applied load is distributed unevenly along the beams and the plate will remain in contact with them only along a middle segment in each one extending from mid span each way to some points PP on each beam where the distributed vertical pressure will have zero intensity. Starting from any one of the points P the intensity of the load applied to the beam will therefore increase from zero at P to a maximum between P and mid span. There will be eight of these points P, two on each side beam, and radial sections drawn from the center O to each of them will be sections of zero vertical shear, in the panel. Any other vertical sections whatever thru O will also be sections of zero vertical shear.

But the area between any such line OP and the nearest median line XX or YY, is one in which the total vertical shear on any section perpendicular to that median line amounts to $W/8$ on that part of the section between *OP* and the median line. The mean intensity of this vertical shear decreases proportionately to its distance from *O*, because the constant total amount of the shear on each vertical section within the triangle *POP* amounting to $\frac{1}{4}W$ is spread over a length of section which increases uniformly with its distance from *O*. It thus appears that the upward reaction that the segment *PP* of the beam *ad* exerts upon the plate is $\frac{1}{4}W$. Its moment about *YY* amounting to $WL/8$ is a bending moment. This moment comprises only one half of $\frac{1}{4}WL$, the moment about *YY* due to the reactions at *a* and *d*, and is resisted by bending resistance in the plate itself.

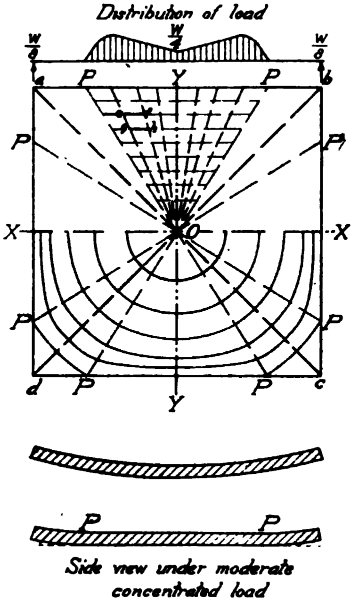

Fig. 98

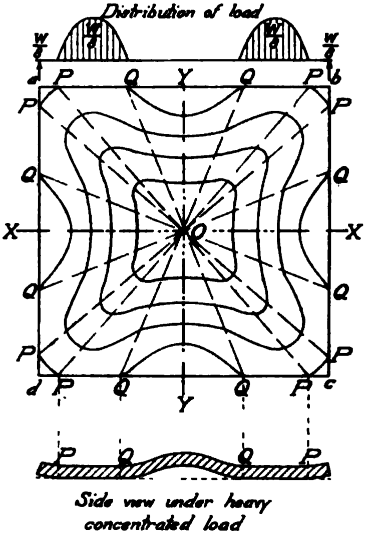

Fig. 99

It appears therefore that while $\frac{1}{4}W$ is transmitted to *a* and *d* together, and $\frac{1}{4}W$ by way of the beam *ad*, $W/8$ is transmitted to *a* by way of the beam *ab*, and $W/8$ to *d* by way of *cd*. Consequently the load $W/8$ which is carried by vertical shears on sections parallel to *ab* and applied to the beam *ab* along the segment *PY* of it between *a* and *Y* has a center of application between *P* and mid span, so that the moment applied to bend the beam is less than $WL/16$ by reason of the fact that its arm is less than $\frac{1}{2}L$. The amount by which the moment resisted by the beam is less than $WL/16$ is that of the twisting moment in the slab about YY called into play by applying the load $W/8$ which originates at O to the beam *ab* at a distance from *a* less than $L/2$. This twisting moment and the bending moment in the beam *ab* together amount to $WL/16$.

The importance that attaches to the present discussion of the action of a concentrated load on a square plate arises from the fact that it is here shown how the load is distributed and applied to the side beams by means of bending and twisting moments in the plate, and that the distribution of the load along the supporting beams is due solely to the twisting moments, which as will be seen later, are also principally responsible for the warping of the plate. It is further shown that whatever may be the magnitude of the twisting moments they reduce by just so much the total amount of the applied bending moment below the statical moment of the applied forces due to the loads. Hence it is evident that if twisting moments are called into play in distributing a single concentrated load upon the supports, they are also necessarily called into play by any distribution of loads, uniform or otherwise, since these may be regarded as made up of a combination of concentrated loads placed more or less closely together.

The twisting moment in that part of a section parallel to *ab* which is on one side of YY is resisted and held in equilibrium by an equal and opposite twisting moment on that part of the section on the other side of YY thru horizontal shearing resistances of opposite signs above and below the neutral surface. These horizontal shears are zero and change sign at YY and increase in intensity to OP where they are greatest. They are also resisted by those portions of the same sections between OP and the outside edges of the plate. It is the deformation produced in these last mentioned portions by the horizontal shears that is the primary cause of the curling up of the plate near the corners.

The magnitude of the twist about an axis lying in an initially flat plate is measured by the difference of slope produced between the tangents to two normal sections of the neutral surface, which are made by planes perpendicular to the given axis drawn at a distance of one unit apart along the axis.

Now the distribution of the load $\frac{1}{4}W$ which is carried to *ab* and applied along PP by the vertical shears on sections parallel to PP may be represented by drawing a series of radial sections of zero shear between OY and OP such that they cut off distances along PY

to which equal shears are applied. The segments are shortest where the intensity of the beam load is greatest and the segments become longer the nearer they are to P. The twisting moment about OY is formed by elements due to vertical shears which are shifted parallel to ab in transmitting an element of the load toward ab. For example the element of vertical shear acting on ef is equal and opposite to that on gh. The product of that element of shear by the perpendicular distance between ef and gh is an element of bending moment about XX, but the product of it by the distance it is shifted parallel to XX is an element of twisting moment about YY. It is evident that the twisting moment vanishes at YY on sections perpendicular to YY, and reaches a maximum of intensity at PO in such sections.

The twisting moment applied to the faces ef and gh tends to produce right hand distortion or angular motion along OY from O toward ab with horizontal shears of the same signs in prolongations of ef and gh toward ad. Such a distortion makes the prolongation of gh toward OY slope downward and makes the prolongation toward ab slope upward so as to raise the plate off from the beams near a. The slopes produced in sections near to ad and parallel to it also cooperate to the same effect.

The contour lines in this plate are shown in Fig. 98, where they differ from those of a circular plate carrying a concentrated load at its center and supported around its circumference in that the contour lines near the sides are flattened out so those that are nearest become almost parallel to the sides near mid span. Those further away from the sides are not so much changed from a circular form. The deflection curve for a mid section is shown immediately below the plan view and the side elevation after bending below that.

The foregoing analysis is applicable to plates having a thickness of the order of 1/40th of the span. For a plate whose thickness is of the order of 1/400th of the span length, this analysis would not be applicable. The difference in the mode of operation of a very thin plate of homogeneous material and a thicker plate under a concentrated load at the center brings out clearly the fundamental difference between plate action and beam action.

Support a sheet of celluloid 1/16″ thick, 12″ square, freely along its edges at about ¼″ in from the edge. Set a block 2″ in diameter in the center of the plate and apply a gradually increasing load on this block. Under a load of 6 or 8 pounds it will be found that the corners of the plate rise slightly from the supports and the contours are approximately those shown in the preceding figure for Type III floor. When this load is gradually increased to 16 or 20 pounds, a sudden change in the phenomena is observed at some intermediate load, where the plate no longer remains in contact with the supports at and near the middle of the sides but rises from the supports at these points as much as $\frac{1}{4}''$ or $\frac{3}{8}''$ as the load is increased as indicated in Fig. 99. The contours, previously concave toward the center of the panel, now become convex toward the center opposite the middle of the sides, and become star shaped outward along the diagonals. When with increase of load the plate commences to curl up along its median lines at the support the deflection at the load suddenly increases, illustrating the phenomena before explained that as the contours change from convex outward to a form alternately concave and convex or star shaped the circumferential resistance is thereby destroyed and the mode of operation changed. Coaction of circular and radial stress then ceases and deflection increases suddenly with this change. By curling up at the middle of the sides the radial resistance of the plate is increased since the moment of inertia of a corrugation is greater than that of the plane plate. Nevertheless, by the removal of the cooperation of circular ring stress with the radial the deflection has increased although the corrugations formed in the plate have increased the radial resistance to bending. Twisting moments also have increased at the same time, since they throw the load carried by the side supports nearer to the corners of the panel and distribute it as shown at the top of Fig. 99. No load, of course, can be brought upon a side support at points at and near the middle of a side where the plate has risen from it, and each half length of a side beam has a loaded segment *PQ* at and near each of its quarter points. The triangular area *QOQ* has zero vertical shear in all sections parallel to *QQ*. The only vertical shears are in the eight triangles *QOP*.

The action of such a plate shows the absurdity of attempting to treat a flat plate supported on four sides by beam strip theory with strips parallel to the sides since the load not only does not travel to the supports in accordance with that theory but is carried in a large measure towards the corners of the plate by twisting moments induced in the plate by double bending.

The peculiar phenomenon of unstable equilibrium reached under a certain load which causes the plate to rise suddenly from the supports at the center of the sides and its deflection to increase suddenly as it rises from those supports, indicates that the twisting moment, because of the intensity of the load, has risen to such a magnitude that the plate is bent thereby and rises from the support, an action which as just stated illustrates forcibly the inapplicability of the beam strip theory with strips parallel to the sides, a theory which has been incorrectly adopted in many building codes and endorsed by the Joint Committee of the Am. Soc. C. E. etc.

If the single load upon the celluloid plate be now removed and four loads be placed one on each of the four quarter points of the span on the median lines the plate will carry a much heavier total load than before without buckling up from the middle of the side supports at all. If, however, the points of application of the load be at the four quarter points of the diagonals, instead the same phenomena which occurred with the central load are observed. Under a uniformly distributed load, no curling up of the corners nor apparent inequality in distribution of the load along the sides of the plate has been observed.

The star shaped contours before described are illustrated in Fig. 99. Also an elevation of the edge of the plate as it curled up from its side supports.

5. Distribution of Reactions Determined by the Direction of Lines of Zero Shear. In uniform plates vertical shearing forces travel in straight lines toward their supports in directions determined by the most rigid parts of the structure considered with reference to their resistance to shearing and twisting. The twisting moments brought into action by the restraint of the plate by its supports takes effect by changing the direction of travel of the load and they thereby alter the distribution of the reaction of the load.

Consider the Type I slab, Fig 100, under beam curvature, in which the contours are straight lines drawn in the figure at the left. If we draw lines aa and $a'a'$ parallel to XX which is normal to the supports we have a beam strip between the planes aa and $a'a'$ along the sides of which there is zero shear and these lines of zero shear are normal to the straight line contours parallel to YY. There is only one other line of zero shear and that is along YY which is coincident with the contour itself and that one contour is the only element which is coincident with the line of zero shear. All other lines of zero shear are normal to the contours. This is typical of beam action and cylindrical curvature.

Because all other lines of zero shear make symmetrical angles with YY it is the axis of zero shear for a type I slab and where there is cylindrical bending in but one direction there can be but one axis of zero shear.

YY and XX, Figs. 101 and 102 in types III and IV are similar axes of zero shear since other lines of zero shear in adjacent octants on opposite sides of XX and YY make symmetrical angles with them.

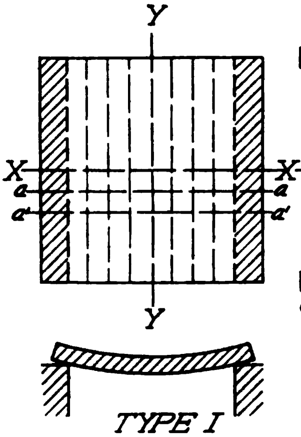

Fig. 100

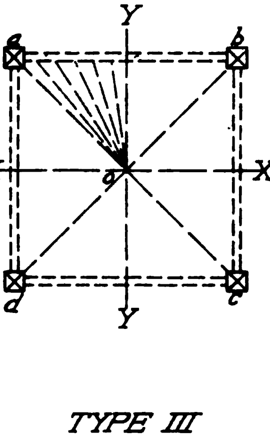

Fig. 101

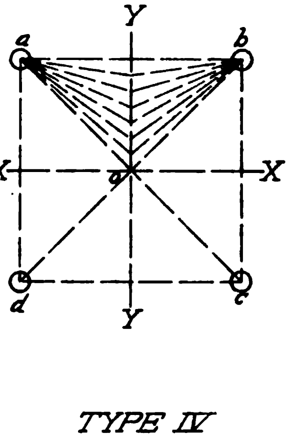

Fig. 102

The intersection of the lines of zero shear at the center forms a focus of zero shear lines in these panels. For a square panel of Type III there is but one focus of zero shear lines while in a rectangular panel there may be more than one as will be shown later.

In dish shaped curvature all axes of zero shear are normal to the contour lines, see Figs. 91 and 92, and one of the principal stresses lies along the contour and the other along the axis of zero shear at their point of intersection. This is true of anticlastic curvature as well except at one point in the center of the saddle shaped areas where the contour intersects the axis of zero shear at an angle which is 45° for simple anticlastic curvature such as that at the middle of either side of a square panel of Type IV floor uniformly reinforced and uniformly loaded.

In section 1 we discussed the localization of reinforcement in respect to contours in a circular flat plate supported on walls and the change produced in contours and mode of operation by localization of reinforcement. In Fig. 103 the same plate is again illustrated locating the axes of zero shear *aa* and *bb* which bisect the angle made by the rigid strips of reinforcement. There are two other axes of zero shear which pass through the center of the heavily reinforced strips. The lines of zero shear make an angle with these axes and for the quadrant in the upper left hand corner the foci of these lines of zero shear will be found on *bb* at points F_1 and F_2. As the rigidity of the strips increases F_1, and F_2 move out along *bb*. As their rigidity decreases these foci approach each other and as the plate becomes more and more nearly uniform in its reinforcement they would finally coincide at the center of the plate.

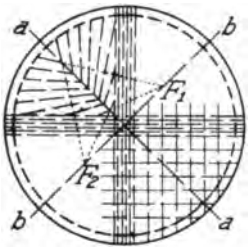

Fig. 103

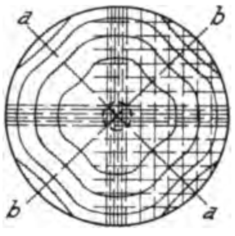

Fig. 104

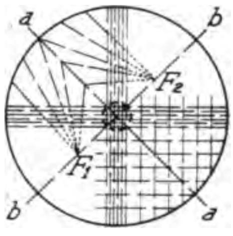

Fig. 105

We will now consider a similar plate similarly reinforced but instead of being supported around its edge consider it supported on a column at its center and monolithic therewith. Under uniform load this would be a circumferential cantilever. Lines *aa* and *bb* would likewise be lines of zero shear and the contour lines would differ from the circular contours of the uniformly reinforced plate Fig. 95, section 1 as shown in Fig. 104, and would resemble those of Fig. 94 rotated through an angle of 45°. The position of the foci of zero shear would be identical with those for the plate supported at its periphery, Fig. 103, with the same reinforcement except that the lines of zero shear would be on opposite sides of the axis *aa* of zero shear as shown in Fig. 105.

6. Distribution of Reactions of Wall and Beam Supports in Rectangular Panels as Determined by the Direction of the Lines of Zero Shear. Employing the principle that vertical shearing forces in uniform plates travel in straight lines toward the supports, let us consider now the case of a rectangular panel supported on four rigid beams or walls. When such a panel is uniformly loaded it is well known* that the plate will form valley lines at 45° from the corners of the plate as shown in Fig. 106. Now the formation of such valley lines induces twisting in the plate and such twisting moments will cause the lines of zero shear at the ends to radiate from the focus F_1 approximately at the intersection of the 45° lines at the ends of the plate, thus distributing the load within the 45° lines uniformly along the supporting walls or rigid beams at the ends. In like manner the rest of the load on the plate will be distributed uniformly along the side walls or beams but this load will be of greater intensity. The focus of the lines of zero shear will lie at F_2 on the axis Y through the center of the plate and the diagonal aF_1. The intensity of the uniform distribution of the load on aeF_1 along ae and the load on $agoF_2$ along ag is readily calculated. (g being the intersection of ab and YY.)

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*This fact is established by all experiments upon such plates under uniform loads.

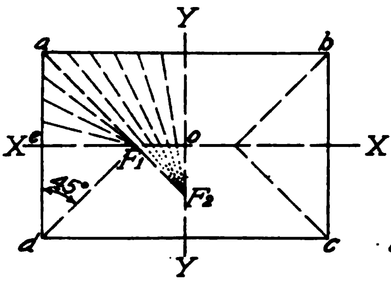

Fig. 106

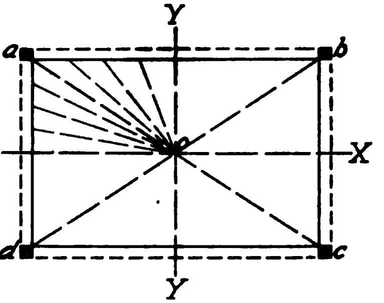

Fig. 107

The magnitude of the twisting moment about XX called into play by the travel of the load W_1 that rests on aeF_1 to ae in such manner that the lines of zero shear radiate from F_1 may be readily calculated as follows:

If $L_1 = ad$, then the statical moment of W_1 acting at its center of gravity about XX is $1/6\ W_1L_1$, and the statical moment of the reaction W_1 uniformly distributed along the beam is $\frac{1}{4}W_1L_1$. The difference of these statical moments $= W_1L_1/12$ is the twisting moment about XX produced by the vertical shears along ae. But $W_1 = 1/8\ wL_1^2$ when $w =$ the load per square unit. Hence the twisting moment about $XX = wL_1^3/96$.

Again, let $L_2 = ab$, then similarly the twisting moment of the load upon $aF_1\ og$ about oY is the difference between that produced by a uniform load on agF_2 and that due to a uniform load on $F_1\ oF_2$ viz.

$$wL_2^3/96 - w(L_2 - L_1)^3/96 \text{ where } oF_1 = \tfrac{1}{2}(L_2 - L_1)$$

These twisting moments are produced by the shifting of the vertical shears which act on sections parallel to a side. These shears represent the travel of the load to the side and they shift along the side at the same time that they progress toward the side. That part of this progress which is directly toward the side produces bending moment, while the shift parallel to a side produces twisting moment. The shift at any point of a section increases in proportion to its distance from the median line, consequently the intensity of the twisting moment increases uniformly on any section parallel to a side from zero at the median line to the diagonal line, where it has twice the mean intensity of the section.

We find the total bending moment applied to the plate about YY by multiplying the load $2W_1 = wL_1^2/4$ that travels to *ad* by the perpendicular distance $1/6\ L_1$ of its center of gravity from *ad*. Hence the bending moment about YY is $wL_1^3/24$.

Similarly the total bending moment applied to the plate about XX is found by multiplying the load $\frac{1}{4}wL_1(2L_2—L_1)$, lying on the trapezoidal area between XX and the side $ab = L_2$ which is transmitted to *ab*, by the perpendicular distance $\frac{L_1(3L_2—2L_1)}{6(2L_2—L_1)}$ from *ab* to the center of gravity of the trapezoid. The total applied bending moment about *ab* is therefore $wL_1^2\ (3L_2—2L_1)/24$. This is one part of the total statical moment $1/8\ wL_1^2L_2$ about XX of the load $\frac{1}{2}wL_1L_2$ of which the other parts are the two twisting moments on the ends, which together amount to $wL_1^3/48$, and the sum of the bending moments in the two end beams, viz. $wL_1^3/16$, which beams each carry a uniformly distributed load of intensity $wL_1/4$. The sum of these three parts added together, gives the total statical moment about XX of the applied load as above stated.

We will now consider the case where the rectangular slab is supported on flexible beams instead of upon rigid beams or walls. As the beams bend or deflect since the slab possesses the same bending resistance in all directions a distribution of the load will occur that will differ from that in case of rigid beams or walls. More load will be thrown upon the shorter beams *ad* and *bc* than was thrown upon the walls *ad* and *bc* in the case of rigid inflexible supports. The bending moment will therefore be increased and the twisting moment will increase with it. The foci F_1 and F_2 of the radiating lines of zero shear will approach each other, F_2 moving along YY toward *o* and F_1 moving along XX toward *o* until when the beams *dc* and *ab* deflect a sufficient amount more than the beams *ad* and *bc* the foci will meet at *o* and a quarter of the load will then be transferred to each of the respective sides and the lines of zero shear will be as represented in Fig. 107.

The bending moment in the plate about YY will be due to the transfer of the load $\frac{1}{4}wL_1L_2$ which lies inside the triangular area *aod* to the side *ad* a distance $1/6\ L_2$ and will amount to $wL_1L_2^2/24$. The twisting moment about YY due to shifting a total load of the same amount parallel to *ab* and *cd* through a distance $L_2/12$ is $wL_1L_2^2/48$ which is one half as great as the bending moment in the panel about YY. The sum of these moments, viz. $wL_1L_2^2/16$, is the total applied moment in the plate about YY. The total statical moment about YY of the panel load is $1/8\ wL_1L_2^2$ one half of which is held in equilibrium by the resistance of the combined bending and twisting moment about YY, while the other half is borne by the moment in the side beams *ab* and *cd* each of which carries a uniformly distributed load of intensity $\frac{1}{4}wL_1$, thus offering a total bending moment in each beam of $wL_1L_2^2/32$ due to the load in this panel.

If these results in Fig. 107 be compared with those in Fig. 106 we notice that the movement of the foci F_1 and F_2 to the panel center *o* has had the effect of transmitting a smaller part of the total panel load to the longer sides and a larger part to the shorter sides. This has made the bending moment in the plate about YY larger in Fig. 107 than in Fig. 106, but has made that about XX smaller in Fig. 107 than in Fig. 106, while the statical moments are unchanged, and further the bending moment about YY in Fig. 107 is larger than that about XX, while the opposite is true in Fig. 106. There is consequently some intermediate degree of beam rigidity between these two cases where the bending moments in the plate will be the same about XX as about YY in which case the foci will be situated somewhat nearer *o* than in Fig. 106. In such case somewhat more than one quarter of the load will be carried to each of the longer sides.

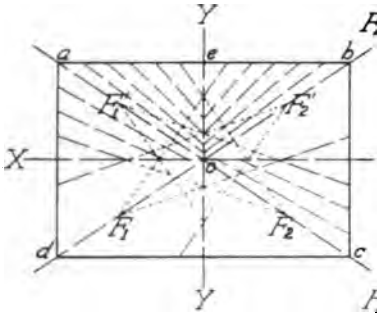

Fig. 108

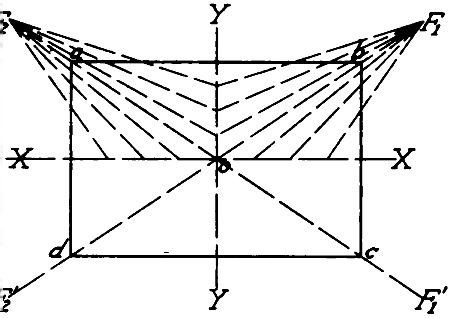

Fig. 109

Consider a still further reduction in the rigidity of the side beams *ab* and *cd* in Fig. 108. The focus of the lines of zero shear which was at *o* in Fig. 107 will move for each quadrant away from *o* along the diagonals of the panel to positions F_1 and F_2 for two quadrants above XX and to positions F_1' and F_2' for the quadrants below XX. Thus the outside lines radiating from the foci to points at mid span on the sides would cross each other thus forming a star with six points.

Thus Fig. 104 with four foci represents what occurs in practice with sufficient accuracy, but cases may be supposed where the rigidity of the side beams *ad* and *cd* is very different from that of the end beams *ad* and *bd*. In such cases the focus from which radiating lines are drawn to the end of a quadrant would be at a different distance from *o* along the diagonal from the focus from which lines are drawn for the adjacent side of that quadrant.

Consider now a further reduction in the relative rigidity of the beams *ab* and *ad*, illustrated in Fig. 109. There would be a further movement of F_1 and F_2 along the diagonals which movement would become very rapid indeed with reduction in the rigidity of the beams until the lines of zero shear would become parallel to each other and F_1 and F_2 had moved along the diagonal to infinite distances. Then with a slight further reduction in the rigidity of the beams the lines of zero shear would not remain parallel but would make an angle with each other in the other direction and F_1 and F_2 would then reappear at an infinite distance on the opposite side of the center *o* and then gradually move backward toward *a* and *c* along the diagonals. They thus approach the positions of the lines of zero shear in the flat slab of uniform moment of inertia on corner supports. Such a distribution of lines of zero shear as is seen in Fig. 109 would occur in the column supported flat plate when additional material is placed in the center of a direct belt or along column lines, thus increasing the moment of inertia in that portion of the panel above that along XX and YY through the middle of the panel.

Various methods have been proposed at different times for attempting to obtain a square panel of area L^2 and supported on side beams which shall be practically equivalent to a rectangular panel of size L_1 by L_2 when both carry the same load W. If the load per square unit as well as the total load be the same in both, then $L^2 = L_1L_2$, and that is frequently assumed to give the size of an equivalent square panel, in which the side of the square is a geometrical mean between the sides of the rectangle.

Another frequent assumption is that the side of the square is the arithmetical mean of the sides of the rectangle, i. e. $L = \frac{1}{2}(L_1 +$

L_2). But no method proposed so far has been found to give so good an approximation to the side L of a square that is equivalent to a rectangle with sides L_1 and L_2 as the formula proposed by Turner in 1909 as follows:

$$L = \frac{L_1^2 + L_2^2}{L_1 + L_2}$$

This formula is based upon the supposition that the intensity of the load is practically the same at all points of all the sides, viz. $\frac{1}{4}w/(L_1 + L_2)$ and that the moment of inertia of the beams is the same on the long as on the short sides, when they have a depth below the slab of two or three times the thickness of the slab.

Further considerations by which this formula for the side of an equivalent square was arrived at had to do with the impairment of the circumferential action in rectangular panels below that which occurs in square panels, but these considerations will not be given here. However, it is clear from previous discussions of circumferential action that it affords one of the necessary bases for calculating the equivalence of panels, and that to take a geometric or an arithmetic mean is a mere guess based on nothing at all.

This value of L when substituted in the formulas both for moments and deflections in square panels has been found to give the moments and deflections in the eqivalent rectangular panels supported by beams $L/10$ in depth.

7. **Continuity in Plate Action.** In flat slabs of Type IV, column supported, continuity between panels is one of the essentials necessary to render the construction feasible or commercially practical. To effect this the steel is required to be placed at the top over the supports. The remarkable effects of full continuity on the resistance of slabs of Type IV are developed in certain important aspects in the next section. If, however, the steel was all of it placed in the bottom throughout and distributed uniformly or in strips no practical continuity could be said to exist between panels, for the column supports would induce anticlastic curvature by reason of the wide difference in rigidity between the reinforced tension zones at the bottom and the unreinforced tension zones over the supports. The slab would then tend to bend in predominantly cylindrical curvature across the column lines which would tend to form a hip along each diagonal line with tension in the upper surface across the diagonals and compression below but with compression across the top of the median line and tension below throughout. Thus, little or no

cooperation of ring stresses would be possible in such a slab since in the change from tension across the diagonal to compression across the median line there must be some point where the stress would pass through zero and change its sign and consequently no continuous circumferential action could exist.

With a beam supported slab of Type III continuity over the beam supports is of relatively small importance. The rigidity of the beam lines in the Type III slab produce the 45° valleys before discussed and the formation of these valleys limits the negative moment over the beams to a nearly negligible area. The radical difference shown to exist between slabs of Type III and continuous column supported slabs of Type IV is the result of the difference in the magnitude and mode of operation of the twisting moments in the two types. The twist in a Type III slab shifts the shears and moments along the beams or sides of the panels and has an effect to reduce bending moments in the slab only about half as much as in the column supported slab where the twist shifts the shears and moments in a direction normal to the side of the panel.

8. The Elastic Bending of Thin Plates in Synclastic Curvature. The curvature at any point P of a curved surface is most readily measured by finding the radii of curvature of two curved plane sections of the surface made by a pair of planes drawn normal to the surface at P and at right angles to each other. Normal planes are those perpendicular to the surface at P, and they intersect each other in the normal to the surface at P. If R is the radius of curvature of any plane section at the point P, then $1/R$ is defined as its curvature at P.

At every point of a convex surface there must, except in case when the curvature of all of the sections is the same, be some one of the normal sections in which the curvature is the greatest, and also another section in which the curvature is least. These are called principal planes and principal curvatures. According to Euler's Theorem these principal sections lie in normal planes which are at right angles to each other, and further, the sum of the curvatures of any pair of rectangular normal sections whatever, at a given point P is constant, so that in rotating a pair of normal planes that remain perpendicular to each other about the normal the increment of the curvature of either normal section is equal to the decrement of the other, and the sum of the two normal curvatures is equal to that of the principal curvatures.

In his great treatise on the Mathematical Theory of Elasticity* Love, following the original investigations of Gauss, demonstrates on page 473 that when a piece of a thin elastic shell or plate that has a spherical curvature of $1/R$ is deformed by a small bending without stretching, then in the case of initial spherical curvature one principal curvature of the deformed surface will exceed the initial curvature $1/R$ by the same amount as the other will be less than $1/R$. That this is the fact in this case seems evident without following the abstruse analysis of Love which treats the bending of thin curved non-spherical surfaces in general, because if x and y be lines drawn tangent to two rectangular normal sections at P, and the spherical surface be bent slightly about x so as to alter the curvature of the section in the normal plane at right angles to x by alternately increasing and decreasing it, it is evident from symmetry that the curvatures of the section in the normal plane at right angles to y will undergo at the same time alterations of curvature which are equal and opposite to those in the first normal plane, altho this equality does not in general hold true in shells that are not spherical in shape. Nevertheless, the same statement evidently holds true for surfaces of revolution about the normal at P as an axis.

Having established the fact that any slight bending about x of a piece of a thin elastic spherical shell or a thin elastic plate in the form of a surface of revolution about the normal is necessarily accompanied by an equal opposite bending about y in case its neutral surface undergoes no stretching, let us consider the bending moments that must be applied to produce such a deformation.

It is evident from the fundamental equation $EI/R=M$ of the ordinary theory of bending that

$$EId(1/R)=d\ M$$

is the equation which expresses the relation between the increment (or decrement) of curvature $d(1/R)$ and the increment (or decrement) $d\ M$ of the applied moment which produces this change of curvature about x (or y) in which the value of I may be calculated in case of only slight curvature just as in a flat plate. It is evident that in order to produce this kind of deformation it is sufficient to apply one positive and one negative moment increment, each of the same magnitude dM, simultaneously about each of the rectangular axes x and y, respectively. These moments produce elongations and shortenings in the exterior and interior fibers of the plate or shell independently in two directions at right angles to each other in the same manner as in a beam but none whatever are produced in the neutral surface, so that the resistance that is offered to this bending arises from the resistance which the fibers offer to such elongation and shortening. The principal curvatures of the surface after bending will be $1/R + d(1/R)$ and $1/R—d(1/R)$ respectively.

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*Second Edition published at the University Press, Cambridge, England, 1906

It is furthermore to be noted that the same deformation may be otherwise produced by applying the sum total of both moment increments, viz. $2dM$, about either x or y alone, and no moment increment whatever about the other axis. We may review and restate this demonstration as follows:

Consider two thin elastic plates of the same material, and of equal area and of equal thickness, the only difference between them being in their curvatures. Let one have a small initial cylindrical curvature $1/R$ where R is large and the other an equal spherical curvature in all directions. These curvatures may either be natural to the plates in their unstressed condition, or they may be initial elastic deformations. Let dM be an element of bending moment that increases the cylindrical curvature by a given small amount $d(1/R)$, i. e.

$$EId(1/R) = dM$$

just as in an initially flat plate. When this bending moment acts on the plate it changes the initial curvature by the same amount whether the initial curvature was or was not produced by initial stresses, provided the bending does not exceed the limits of elasticity.

But if an element of bending moment dM could be applied to produce elongations and compressions in the fibers perpendicular to a given axis x in the spherical plate and in these fibers alone, it would produce an increment of curvature in sections perpendicular to x equal to that in the cylindrical plate. But dM when so applied to the spherical plate cannot act to do this independently and without hindrance as it is able to do in the cylindrical plate because this increment of curvature $+ d(1/R)$ in planes perpendicular to x is necessarily accompanied by an equal decrement of curvature $—d(1/R)$ in sections perpendicular to y which decrement requires the application either of an additional element of bending moment of the same amount dM about x or of a negative element—dM about y to diminish the curvature in sections perpendicular to y by the amount $—dM(1/R$ with its equal elongations and compressions.

That the action of an element of bending moment $—dM$ about y is equivalent in its action to a numerically equal element $+dM$ about

x follows from the fact that the amount of work stored in the elongations and compressions of the fibers parallel to x is the same as in those parallel to y during alterations of curvature, and whether that work is performed by the bending moment about x or by the bending moment about y, is immaterial. The forces involved in $+dM$ acting about x move thru the same distance as $-dM$ about y, since the deformation $+d(1/R)$ about x is numercially equal to the deformation $-d(1/R)$ about y.

Hence the spherical plate or shell when acted on by an increment of bending moment $+2dM$ about x alone will receive the same increment of curvature $d(1/R)$ about x as an equal cylindrical plate would receive when acted on by an increment dM one half as large, but the spherical shell will also at the same time suffer a decrement of curvature $-d(1/R)$ about y.

Now let us consider a piece of a thin elastic plate or shell which has first been slightly deformed from its initial shape of a surface of revolution about the normal at P, into a surface whose principal curvatures at P are $1/R + d(1/R)$ and $1/R - d(1/R)$ respectively, by the application of a bending moment $= 2dM$ about x and no moment about y. Secondly, let a positive bending moment equal $2dM$ be also applied to it about y in addition to the positive moment $2dM$ already applied about x.

It is thus subjected to equal positive bending moments about both x and y. This will increase the sum of the curvatures in sections at right angles to each other and make the sum have a value greater than it had previous to the application of these moments, and will involve a stretching of the neutral surface, and a resistance due to such stretching besides that due to bending.

In the first deformed state where the moment $2dM$ was applied about x alone opposite alterations of curvature of $+ d(1/R)$ were produced about x and y, and the entire effect of this applied moment was employed in producing bending, but when a positive applied moment $2dM$ about y is superposed upon the foregoing, only part of its effect acts to overcome resistance to bending while part produces stretching of the neutral surface.

The consequence of this is that the application of the additional moment $2dM$ about y while it produces a certain amount of positive curvature about y and of negative curvature about x does not produce so much alteration in those curvatures as it would were $2dM$ not acting about x. From this it is evident that when moments each equal to $2dM$ are simultaneously applied about both x and y to the convex surface whose initial curvatures were both $1/R$, then the equal resulting curvatures of the stretched surface will be less than $1/R + d(1/R)$ *i. e.* the elongations of extreme fibers shown by extensometer measurement will be less than half as much as the moment $2dM$ would produce in cylindrical bending of a beam or flat plate.

But a thin flat plate or slab after double curvature has begun to take place in it is a thin shell in which the normal curvatures affect each other in the manner just investigated, and require in order to produce given changes of curvature and given surface elongations more than twice the applied moment that would be required in producing cylindrical curvature in the flat plate.

The question that arises in connection with this employment of the fundamental principle of the mutual relation of changes in the principal curvatures of thin shells in its application to flat plates in which double curvature has been produced by bending, is this: How much curvature must be produced in a flat plate to require the application to it of this principle which controls the changes in the curvatures of thin shells. The answer is this: Whenever the double bending is such that the tensions and compressions due to the bending are sufficient to produce actual stresses and strains in the fibers of the plate in directions parallel to the two principal planes of magnitudes approximately equal to each other the plate then acts on the principle of a thin shell and the above demonstration as to a thin shell holds true of the plate; but when the stresses and strains in one direction have magnitudes very different from those at right angles thereto, the mutual effect of changes of curvature is very much reduced.

9. Concrete Slabs in Which Plate Action is Partially Developed. By staggering the columns where the ordinary arrangement would interfere with the desired arrangement of lines of boxes, shelving and walls an economical column spacing may be adopted. This diagonal arrangement of columns is merely placing each column opposite the mid point between columns of adjacent rows.

The first concrete structure of this type was the Bostwick Braun wholesale hardware building at Toledo, built in 1907, on the Mushroom system. The column spacing was 17′0″ centers on diagonal lines giving rows 12′0″ center to center on lines parallel to the walls, thus permitting lines of boxes to be built from column to column parallel to either wall and still only 12′0″ center to center. The column spacing 17′0″ on diagonal lines was more economical than 12′ by 12′

or 12′ by 24′ or 24′ by 24′, one of which would be required to give the same box or shelf arrangement without loss of space, allowing the columns to go through the boxing if a strictly rectangular arrangement of columns were adopted.

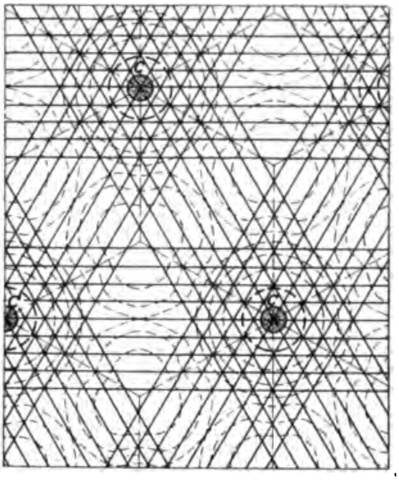


Fig. 110

Mr. David W. Morrow of Cleveland in U. S. Patent 1,064,850 discloses a modification of the Turner arrangement of the Bostwick Braun Building in which the columns are likewise staggered but arranged in triangular formation instead of rectangular, as shown in Fig. 110. In this Figure the full lines represent the reinforcement, with the columns at *C*, *C*, cross hatched above the slab. The lines of slab reinforcement intersect at an angle of 60°. A circumferential cantilever of substantially uniform strength is formed in the slab over the column heads but the suspended plate between the columns is not uniformly reinforced. The lines of zero shear radiate from the columns and there are six axes of zero shear which are shown by dotted lines. These axes form hexagonal figures enclosing a panel area supported by three columns which figure is subdivided by the axes of zero shear into three diamonds. The axes of zero shear are not continued beyond their intersection between the columns as would be necessary were all lines of zero shear under uniform load drawn. The contours, it will be observed are circular in the cantilever portion of the slab but are not circular in the suspended span portion where the triangular areas of the massed reinforcement tend to cause anticlastic or scallop shaped curvature. The effect of such curvature has been discussed in Section 2 of this chapter.

This construction therefore presents a continuous concrete plate that develops little if any plate action in the suspended span portion, but full plate action in the circumferential cantilever portion.

The zero shear lines under uniform loads radiate from the columns to the center of the diamond between the columns in a manner similar to that discussed for the column supported square panel in Chapter

III. The twisting moments normal to the column lines shift the moment and shears toward the column lines in the travel of the load toward the support, as in the square panel before treated.

The lack of uniform distribution of the reinforcement about the center of the suspended span necessitates the treatment of this portion of the slab on substantially beam theory of action and necessitates heavier reinforcement or a thicker slab than is required where uniformity of distribution produces synclastic curvature.

Another form of reinforcement embodying the defect just discussed in a more aggravated form is presented in the illustration of the S-M-I system in the Engineers Handbook on concrete by Hool, Johnson and Hollister, page 474, reproduced in Fig. 111 in which the cantilever reinforcement consists of circumferential rings and radial reinforcement similar to various forms of the prior Turner construction as described in patent No. 1,003,384 in which Claim 24 reads:

> **"The combination of a concrete column comprising a series of vertical bars, means connecting said bars and forming a framework therewith, a framework at the top of the column extending laterally outward in different directions and consisting of a series of radial members and concentric circular members supported by the radial members, and a concrete slab supported by the columns and integral with the column."**

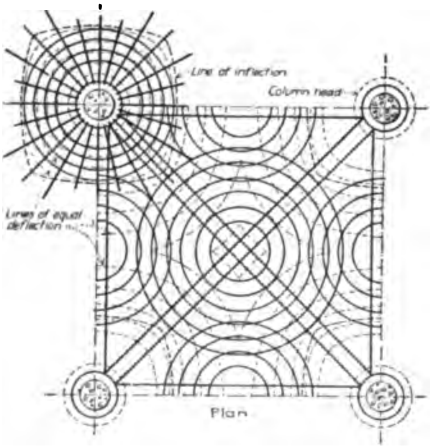


Fig. 111

The radial bars in the S-M-I construction, are hairpinned to extend outward radially in the bottom as well as the top. The contours appear in this figure as light dotted lines. The convex direction of the contours toward the center of the panel which is undoubtedly correct, is brought about by the lack of uniformity in distribution of the diagonal rods relative to the ring reinforcement in the suspended span portion about the panel center, produces anticlastic curvature as shown by the contours and destroys imitation of plate action in the center of the plate under the principles explained in Section 2 of this Chapter. But the pointed apices of the contours have no justification by scientific measurements. This structure therefore requires either additional thickness of the slab or an excessive amount of steel and the additional expense of fabricating the rings for the central part of the slab.

The statement that this is the best known if not the only type of circumferential system which appears on page 471 of this handbook, shows that the authors of the handbook were not familiar with the prior patent above referred to nor with the disclosures of U. S. Patents No. 1,036,384, August 1912, of patent 1,072,532 of September 1913, and others, all of which disclose circumferential members but obviate that concentration of metal which produces anticlastic curvature. Had these writers taken occasion to state the fact that this is the only circumferential system in which circumferential rods are introduced to resist longitudinal stress on direct column lines where economy would indicate straight rods they would have called attention to the most outstanding peculiarity of the system.

Circumferential reinforcement in the form of flat spirals combined with radial rods has certain advantages in areas of synclastic curvature, for it enables the designer to secure a fine grained mat of crossed rods while avoiding the piling up of too great a thickness of metal where it is massed over the column.

In the Engineering Record, February 19, 1916, certain tests of cantilevers with strip reinforcements are described. Some of them had radial and ring rods in crossed relation for reinforcement and others had narrow strips of rods crossed at 90°. From these tests it appears that the use of rings without a proper proportion of radial bars is inefficient which is in accordance with the principles we have previously demonstrated. It appears further that the mat of crossed rods must cover the outer portion of the cantilever area to get effective coaction of metal and concrete to resist jointly the radial and circumferential stresses. In tests 7 and 8 for example, the crossed rods at 90° extending outside the cap, cover a square in which the ring shaped area within a circle tangent to the sides of the square and between that circle and the column cap where the reinforcement was effective, was about one fifth as great as the area covered by the crossed radial and ring rods of tests 1, 2, 3 and 4. There was thus little coaction of radial and circumferential stresses in specimens 7 and 8. Hence the comparison of these cases with the mat of crossed radial and ring rods covering the entire area was no criterion whatever of the relative advantage of the respective kinds of mat where the crossed rods cover the outer portion of the cantilever in both cases.

In the same article a further comparison was made of tests of flat slab panels in which the loading was differently distributed. The slab tested was 20′ square and over-hung four columns spaced twelve feet center to center. The load was approximately uniform with a slight increase at the center of the panel and this test was compared with another previously published, in which the load was very largely concentrated at mid span. A computation of the applied moment on the median lines of the slabs in the cases compared reveals the fact that this moment is five times as great in the case of the slab with unbalanced load as in the case of the slab with balanced load which was treated as its equivalent, a fact that renders the conclusion as faulty as the premises from which it was drawn.

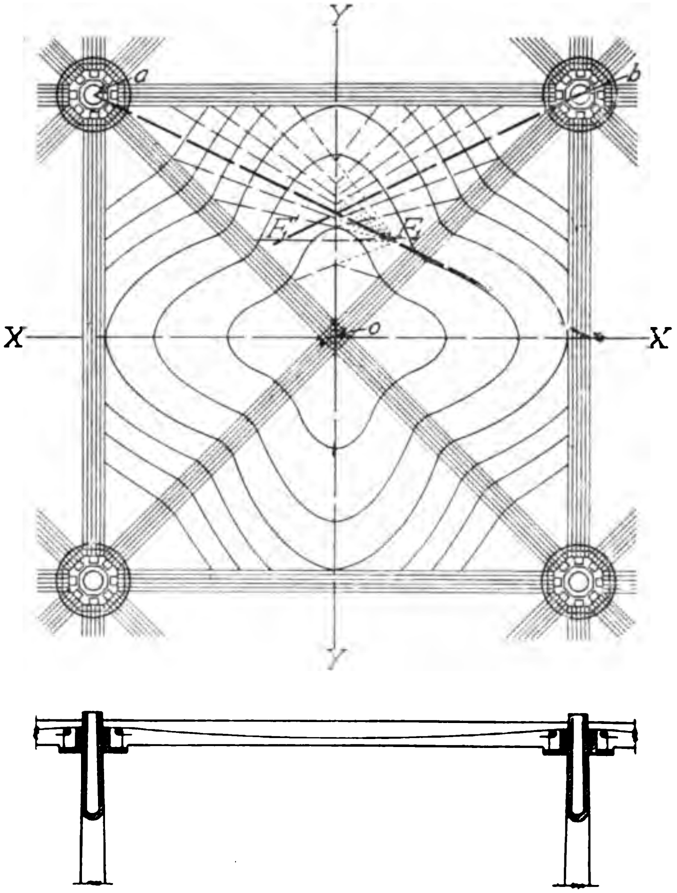


Fig. 112
From Austrian Patent to Speer Oct. 1909.

10. Concrete Floors That Do Not Develop Plate Action. Again, consider a panel of a flat plate supported on columns but reinforced with belts of rods lengthwise, crosswise and diagonally from column to column as shown in Fig. 112, then the rigidity of the respective strips *ab*, *bo* and *ao* would be such that the strip *ao* being along a diagonal which is longer than a side would deflect about one and a half times as much as *ab*, the direct span. The greater rigidity of the strips *ab*, *ao* and *bo* above that of a plate without reinforcement in the triangle *abo* would cause valley lines that would nearly bisect the angles between the successive lines of greatest rigidity at each support. The foci of the lines of zero shear are moved from *o* along the lines bisecting the apices of the triangle *abo* to new positions F_1 and F_1' and these lines would radiate therefrom. All contour lines would be normal to the lines of symmetry, viz. to the diagonals and median lines of the panel, and the contours would be four pointed star-like figures approaching that of a four leaf clover about the center. There would be eight foci of lines of zero shear against four as found in a panel of the Mushroom type slab, and sixteen axes of zero shear against eight in the Mushroom type.

The change in curvature in the contours from concave at all points toward the center of the panel to alternately concave and convex is conclusive proof that there can be in such a plate no cooperation of circular and radial stresses since the circular or ring stress passes through zero with the change of the curvature of the contours from concave to convex toward the center of the plate. Thus the radical difference in mode of operation is graphically disclosed by the radical difference in the form of the contours as compared with those of the plate reinforced to imitate the action of the homogeneous plate.

In order to determine the bending moments resisted by the beam strips in Fig. 112 we must consider the distribution of the load upon them as determined by the lines of zero shear. Assume for simplicity that the foci F_1 and F_1' fall upon the diagonals, let *e* designate the middle of *ab* and call the point *c* where the line aF_1 intersects bF_1' on the median line YY, then the load situated upon the area *aco* is transmitted to the beam *ao*, while the load situated upon *ace* is transmitted to *ae*, the total load upon *ao* being usually greater than that upon *ae*, while the intensity of distribution increases nearly uniformly from the mid spans *o* and *e* respectively to the support *a*.

Such distribution is brought about by twisting moments in the plate about axes perpendicular to the beams, and they reduce by so much the bending moments in the beam strips below the statical moments of the loads they would otherwise be called on to resist.

For example, the statical moment of the load $W/16$ on *ace* about YY is at least $WL/48$, but when the distribution of this load increases uniformly from e to a, the bending moment is only half that amount, the rest of the statical moment being resisted by the twisting moment in *ace* about YY.

It appears therefore that altho the arrangement of reinforcement in four directions in strips with open areas between as in Fig. 112. affects some reduction in bending by reason of the twisting moments between the strips, to less than the statical moment of the load, by the amount of the twisting moment, it nevertheless does not secure the advantage of reduction of steel stresses through cooperation of radial and circumferential action such as is secured in a plate where the rigidities permit the continuously concave contours of synclastic curvature.

The foregoing principles may be illustrated by the details of a test made on two slabs shown in the Figs. 113-114 and 115. This test was undertaken to investigate the diverse effect on the amount of deflection and mode of operation of steel arranged to cover the area as cantilever and suspended span reinforcement on the one hand and arranged in the bottom as flat arch ties with open areas on the other hand.

The test slab with the narrow belts and open areas has an arrangement of steel covering the areas shown reinforced in the Norcross flat arch patent No. 698,542 and was freely supported on the column caps.

The Mushroom slab was rigidly built into the columns following the usual custom. The freely supported slab, because the steel was in the bottom, was given but a small overhang over the supports since the concrete would have been liable to crack under the weight of any one walking on it had it over-hung any farther, and in order to make the test most favorable this arrangement with the steel in the bottom the load was applied in the form of a cross. It will be evident that had the load been applied uniformly instead of concentrated, that the over-hanging portions of the freely supported slab because of the bottom position of the steel would have broken off at a very low intensity of loading since the concrete in the top would obviously furnish but negligible resistance. In fact the overhanging ends broke off when about one hundred pounds per foot was later placed thereon.

On the other hand, had the load been applied uniformly to the Mushroom slab, the balanced cantilever loading would have greatly reduced the moment at mid span so that the slab would have carried two or three times the load with no greater stress. For that reason the test was most favorable for the flat arch slab with the steel in the bottom throughout.

Fig. 118 shows load 4 applied to the Mushroom slab with the areas marked on plan view of the reinforcement. Fig. 117 indicates the position of the load tabulated in the accompanying tables.

In the flat arch slab the embedment of steel under the foregoing principles imparts rigidity to the slab along the reinforced areas thereby producing saddle back curvature across the diagonal accompanied by tension in the top and ultimate cracking in the concrete along the diagonal as was observed. Such saddle back curvature produces star shaped contours similar to those illustrated in Fig. 112. Such star shaped contours indicate that in circumferential lines about the panel center there are points where the stress is zero and hence no cooperation of radial and circular stresses could occur. Accordingly the carrying capacity of the flat arch slab

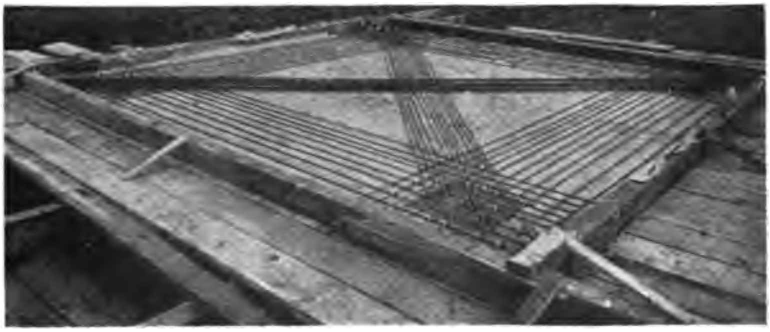
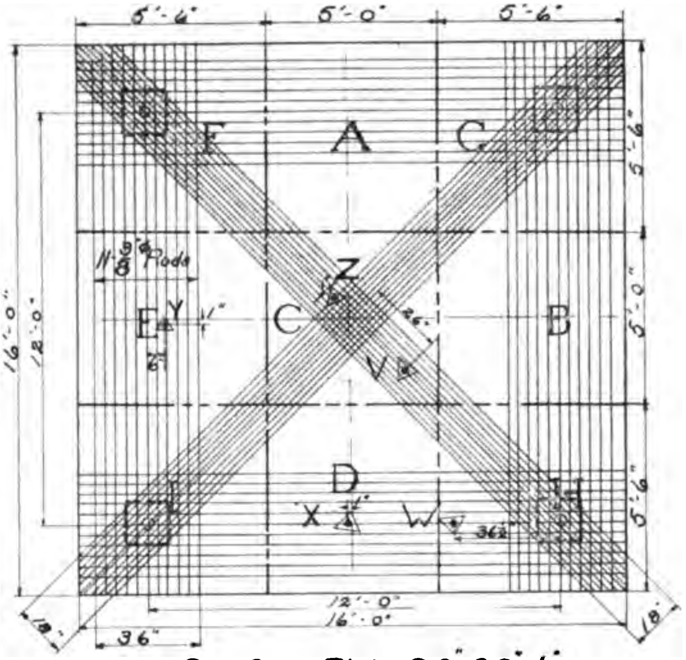


Fig. 113



Col. Cap Plate 20"x20"x½"
Belts 11-⅜"φ each way

Fig. 114. Flat Arch Type Slab.

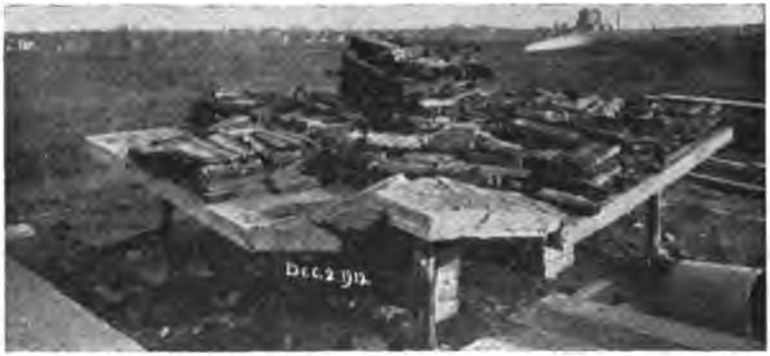


Fig. 115. Collapse of Flat Arch Slab.

reinforced in four directions can be readily accounted for by beam theory. Whatever twisting moments are induced merely distribute the load along the diagonals to some extent. Moreover, from the principles before developed governing the distribution of bond shear stress it is obvious that indirect stresses would be unable to operate across the wide unreinforced areas and for that reason no circumferential action by bond stress would be possible. Only linear arch tie action would be possible.

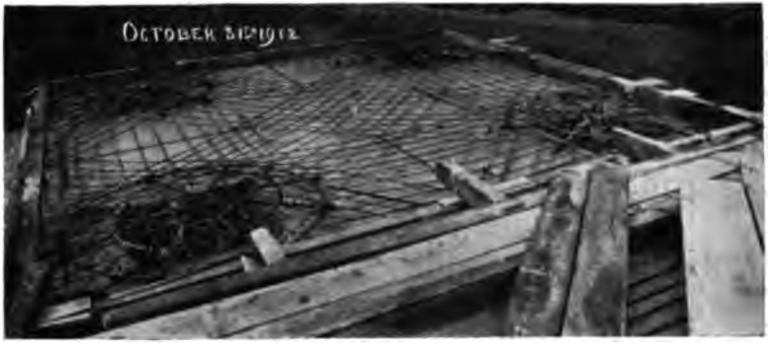


Fig. 116. Reinforcement of Mushroom Slab.

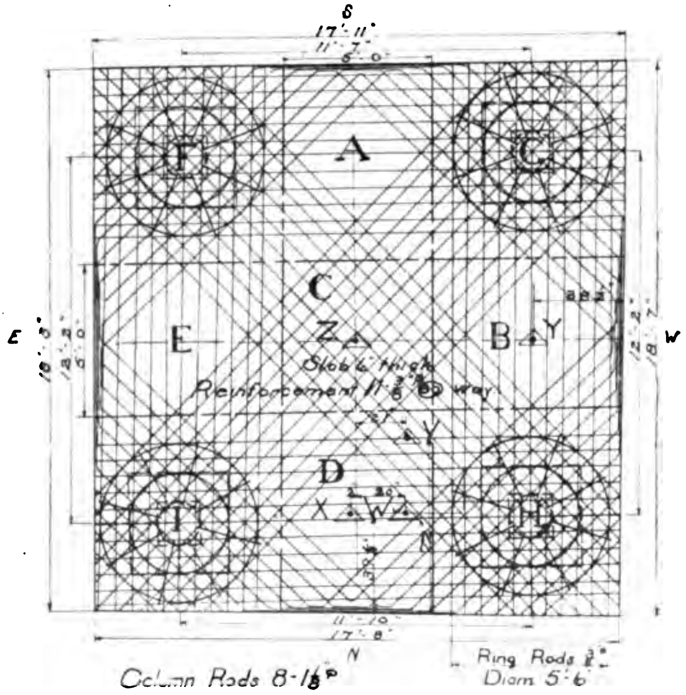


Fig. 117. Mushroom Slab.

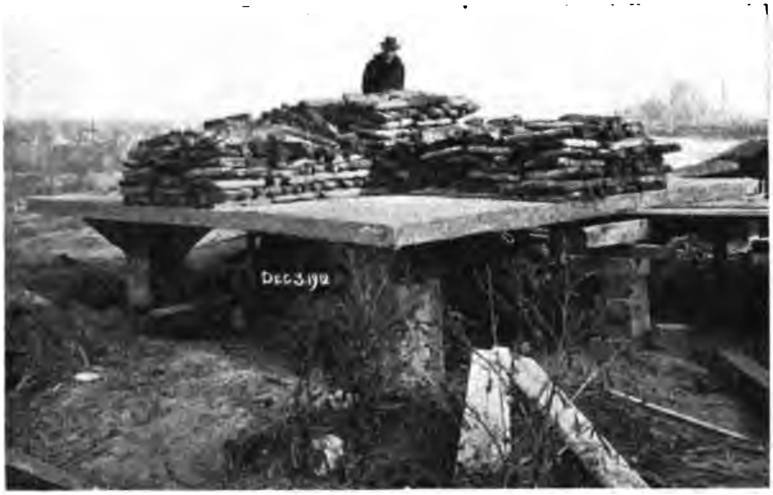


Fig. 118. Mushroom Slab, Load 4.

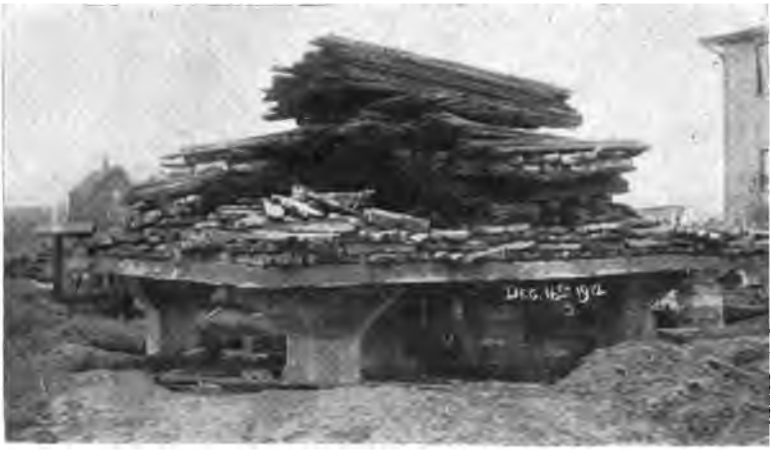


Fig. 119. Mushroom Slab, Load 9.

Area	Load Per Sq. ft. lbs.	Total on Area	Area	Load Per Sq. ft. lbs.	Total on Area
A.	899	29,222	F.	250	10,560
B.	899	29,222	G.	250	10,560
C.	2689	67,236	H.	250	10,560
D.	899	29,222	I.	250	20,560
E.	899	29,222	Total on Slab	Load 9	226,365

Table 3. Loads on Flat Arch Slab in Pounds

No.	1		2		3		4	
Areas	per sq. ft.	Total	per sq. ft.	Total	per sq. ft.	Total	per sq. ft.	Total
A	114.1	3138	228.7	6288	228.7	6288	228.7	6288
B	114.1	3138	228.7	6288	228.7	6288	228.7	6288
C	114.1	2852	240.1	6002	456.8	11420	456.8	11420
D	114.1	3138	228.7	6288	228.7	6288	228.7	6288
E	114.1	3138	228.7	6288	228.7	6288	228.7	6288
F							250	7560
G			Corner	Areas Not	Loaded		250	7560
H							250	7560
I							250	7560
Slab	60	15404	121	31154	143	36572	276	66812

Table 5. Loads on Mushroom Slab in Pounds

Load	1		2		3		4		5	
Area	per sq. ft.	Total	per sq. ft.	Total	per sq. ft.	Total	per sq. ft.	Total	per sq. ft.	Total
A	100.8	3276	201.6	6552	201.6	6552	425.3	13822	625.3	20322
B	100.8	3276	201.6	6552	201.6	6552	425.3	13822	625.3	20322
C	100.8	2532	201.6	5040	418.3	10458	861.8	21546	1261.8	31546
D	100.8	3276	201.6	6552	201.6	6552	425.3	13822	625.3	20322
E	100.8	3276	201.6	6552	201.6	6552	425.3	13822	625.3	20322
				Corner	areas no	t loaded				
Slab	48.2	15624	96.4	31248	113	36666	237	76835	348	112836
Inside the panel		9576		19152		24570		51317		75318
On the overhang		6048		12096		12096		25518		37578

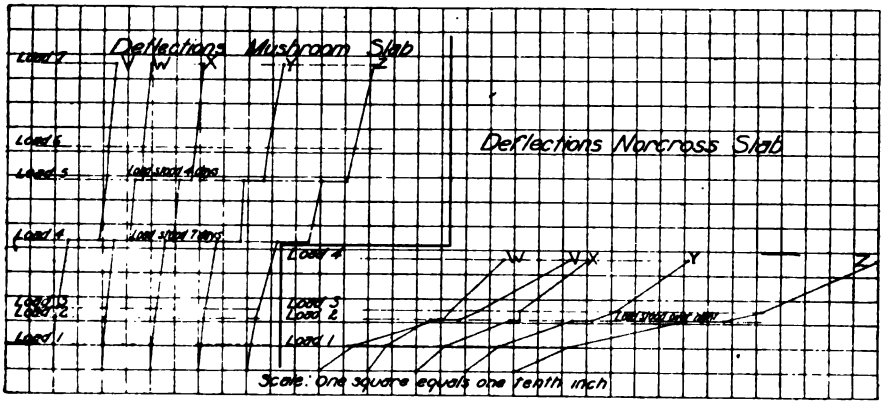


Fig. 120. Comparative Deflections of Flat Arch and Mushroom Slabs Disclosing Difference in Mode of Operation.

The graphical representation of the experimental observations in the deflections at the points V, W, X, Y, Z, of the two slabs found in Fig. 120, shows in a striking manner how small the loads and how great the deflections were in the Flat Arch Type slab on the one hand, and how large the loads and how small the deflections were in the Mushroom slab on the other hand.

The following table shows the marked difference between the ratio of deflection and steel stress, because of the difference in the operation of bond shear stresses in these two slabs. The effect in detail of the rigidity of the column supports will be investigated as a general problem later.

TABLE

		Load		Steel Stress			
Type	On each Area A B E D	On Area C	Total	Average on belts at mid span	Ratio	Deflection	Ratio
Flat Arch....	3138	2852	15404	7690		.2207	
Mushroom....	3276	2532	15624	643	12	.017	12.8
Flat Arch....	6288	6002	31154	27133		.707	
Mushroom....	6552	5040	31248	1154	23.5	.0353	20.
Flat Arch....	6288	11420	36572	34894		1.020	
Mushroom....	6532	10458	36666	1472	23.4	.0463	22.

That the difference in deportment cannot be accounted for by difference in the restraint of the supports is apparent upon the application of the law of moment magnitudes. Five times the stiffness in the case of full restraint at the support for the same section at mid span is all that would be possible were there not a different mode of operation brought about by spreading out the same amount of steel so that it covers the area and operates upon the radically different principle of plate action instead of upon the principle of the arch. With the distribution of metal in narrow belts with open areas, anticlastic curvature is produced under flexure, across the diagonal belts of the flat arch type of slab, thereby cutting off all circumferential resistance for the reasons given in Section 2 of this chapter. A plot of the contours and lines of zero shear would be similar to that drawn for the belts lengthwise, crosswise and diagonally, of the Speer patent of 1900.

9. Remarks. From the foregoing discussion of contours and elastic bending in double curvature the error of the customary treatment of flat plates in building codes and theoretical treatises on reinforced concrete becomes apparent. The applied bending moments computed according to the ordinary building ordinances are more than twice as great as the loads actually produce in such plates, and in many instances they permit plates that are too thin relatively to the span length for light loads, while requiring plates excessively thick and lacking in enconomy for heavy loads, as will appear later. City Building Departments have been very slow to use results of actual tests as a basis for their codes. Few of them have made a conscientious endeavor to utilize any sound theory for the formulas they adopt.

The following conclusion may be drawn from the discussion of the elastic bending of the thin plate regarding a wall panel which is supported on a relatively rigid beam on one side and upon columns at two corners. In order to secure bending approaching that of synclastic curvature so that practically the same law of operation holds true with respect to a wall panel that holds true with respect to an interior panel the rigidity along the sides from the wall to the adjacent columns must be increased over that of a similar side in an interior panel to such an extent that the deflection along that side will be nearly a mean between the deflection occurring between the corners supported by columns and the deflection of the beam supported side. When this is done the material about the diagonal center of the panel will operate as a mat or sheet just as is the case in the regular interior panel.

It will be observed that the elastic theory of bending thin plates which shows that the resistance in synclastic curvature is double that of cylindrical curvature can be applied only to fairly uniform distribution of reinforcement and is not applicable where such distribution is lacking as in the case where there are large unreinforced areas.

Any considerable irregularity in the distribution of reinforcement in areas of synclastic or approximately synclastic curvature changes that curvature to anticlastic curvature in those areas thereby changing the mode of operation to approximately that of beam strip action. On the other hand, concentration in the areas of anticlastic curvature, such as the saddle shaped area of the column supported flat plate does not so greatly alter either the mode of operation, the stiffness or the strength, as is the case in the areas of synclastic curvature.

CHAPTER X

THE CONTINUOUS UNIFORM SLAB ON SEPARATED SUPPORTS UNDER UNIFORM LOADING

1. The Vertical Shears. Draw the panel diagonals thereby joining the centers of the supports at the corners of the panels. The total area of the slab will then be subdivided in all its interior portion into diamond shaped areas or rhombs each of which has an area equal to one half that of a panel.

Since the vertical section along each of these diagonals, as well as along each of the sides and median lines of the panels, is a line of zero vertical shear at all points, these diamond shaped areas each constitute a separate beam whose sole supports are at its two ends at the edges of the caps. Mutual restraint exists all along the edges at each diagonal and at the edges of each cap but no vertical shear exists anywhere on the perimeter of a diamond except at the edges of the caps. Each of the diamond shaped beams that radiate from a column cap may be regarded as subdivided into four equal parts by a median line, and a side line of a panel, and is to be treated as part of a continuous beam so restrained as to remain horizontal at the median line and the edges of the caps.

It is self-evident that when the load on each diamond has been thus subdivided into four equal parts each part travels to its support by vertical shears in that area of the diamond on which it rests and the analysis of the slab is identical with the analysis of these diamond shaped beams of which it is composed when each is subjected to a load uniformly distributed over its area and to the restraints just mentioned. This uniformly distributed load constitutes a kind of loading whose amount increases uniformly along the span between column centers from zero at each end to its largest value at mid span. Two separate analyses of this problem are given in the following sections in addition to the statical demonstration already given in Section 23 Chapter III.

The first of these analyses treats loads, total shears and bending moments, and in it no assumption is made other than the single fundamental equation of beam action which states that in transmitting loads to the supports every element of a bending moment about an axis is due to a shift perpendicular to that axis of a shear that is parallel to that axis. The second analysis assumes in addition to the foregoing that every element of a twisting moment about an axis is due to a shift sidewise perpendicularly to that axis of a shear that is also perpendicular to that axis. These assumptions may both be regarded as merely accepted definitions of the inherent nature of bending and twisting moments. It has been thought desirable to insert these several demonstrations of the same result in order to establish beyond any question of doubt a point in the theory of plates and slabs that has apparently been heretofore overlooked and treated erroneously in all current engineering literature.

It should also be said respecting each of these diamond shaped beams that while there can be no possible difference of opinion as to the bending moments about the right cross sections due to transmitting its loading to the supports when subjected to the restraints at its ends, nevertheless, the moments about longitudinal sections have not been regarded with the same unanimity of opinion and they must therefore be here considered more particularly.

In order to do this, suppose the diamond shaped beam were separated into two equal beams by making a longitudinal section along the panel side from the edge of the supporting cap at one end to that at the other end. The diamond shaped area would then afford no resistance whatever to a cross bending moment, so that the statical moment of the load about the panel side would necessarily be resisted by twisting moments alone.

The question therefore is whether the statical moment of the loading upon the diamond shaped area when not so subdivided calls into play bending moments about the line joining the column centers or are they twisting moments. In which ever way that question may be answered in case of an isolated diamond shaped beam that is not separated into two equal parts by a longtidinal section, it is known by experience when this beam is continuous with the rest of the slab, that the statical moments mentioned are resisted almost entirely if not entirely by twisting moments because in many slabs both those that have been tested and those that have not been tested which have no top reinforcement across the panel sides very small cracks only appear from column to column along the sides.

Were the bending moments across the sides of more than negligible size these cracks would increase with the loading and be much larger than they are. The fact is that of the two kinds of resistance to the statical moment about a side, viz. bending and twisting, the more rigid of the two will be the one principally in action. Since the resistance to twisting which consists of horizontal shears calls into play not merely the width of the right sections of the diamond shaped beam but the entire width of the panel as well, and these shears furnish far more rigid resistance to twisting than the resistance of the slab to bending, the statical moments mentioned are resisted in continuous slabs almost entirely by twisting. This result may be taken as a confirmation of the principle of the statical analysis employed in Section 23 Chapter III and in Section 4 of the present chapter.

2. Analytical Investigation of Bending Moments in a Square Panel. Let $OADB$ in Fig. 121 represent one quarter of a square panel of a continuous flat plate or slab carrying a uniform load of w pounds per square unit of area, in which O is at the middle of the panel, D is at the center of one of the corner supports, and L is the distance between column centers. Let $W=wL^2$ be the total panel load. Then taking the origin of coordinates at D let $x=DF$ and $y=DE$ be the coordinates of any point P on the diagonal DO. By reason of symmetry the vertical shear is zero at every point of the diagonal DO and at every point of the four sides of the quarter panel $OADB$. Also $OA=OB=\frac{1}{2}L$.

Disregarding the small part of the load that rests directly upon the column cap, a total vertical shear $=\frac{1}{4}W$ acts in that quarter part of the perimeter of the column cap which lies inside the quarter panel. But in any case the total vertical shear acting at the sides PE and PF of the square area $PEDF$ is the difference between the vertical reaction $\frac{1}{4}W$ at the support D and the load wx^2 on the area $PEDF=x^2$; hence the expression for the total vertical shear at the two edges PF and PE of any area $PEDF$ is

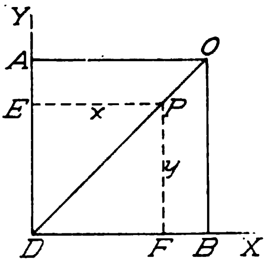


Fig. 121

$$\tfrac{1}{4}W-wx^2=S_x+S_y \qquad (1)$$

in case S_x and S_y represent components of the total vertical force perpendicular to x and y respectively which are transferred by shearing stresses across the perimeter of the area $PEDF$ within a quarter panel area.

This equation (1) is a particular case of the general condition of equilibrium for the vertical forces acting upon any given piece of the plate, since it requires that the algebraic sum of all the applied vertical forces, such as reactions of supports and loads which are applied to it inside of its perimeter, shall be equal and opposite to the sum of all the separate components of the vertical shears applied at the entire perimeter of the piece. Now, whatever size the cap may have, $\frac{1}{4}W$, and wx^2 are the only vertical forces applied inside the permieter of *PEDF* while S_x and S_y are the only separate components of the vertical shears at the perimeter. Since as already stated *DO*, *DA* and *DB* are sections of zero shears, S_x and S_y are shears on *PF* and *PE* respectively. We have thus by symmetry

$$S_x=S_y=\tfrac{1}{2}(\tfrac{1}{4}W-wx^2)=\tfrac{1}{2}(\tfrac{1}{4}W-wy^2)\ldots\ldots\ldots(2)$$

Again, let $\frac{1}{2}M_x$ and $\frac{1}{2}M_y$ be the x and y components distributed across the width of one quarter of the panel of the bending moments due respectively to S_x and S_y. Since S_x and S_y act separately upon sections taken perpendicular to each other they each generate bending moments separately. Hence

$$\tfrac{1}{2}dM_x=S_xdx,\ \text{and}\ \tfrac{1}{2}dM_y=S_ydy\ldots\ldots\ldots(3)$$

in which M_x and M_y refer to moments across the entire panel width.

The fundamental equations (3), since they refer to applied moments as well as shears at right angles to each other treat of shears as well as bending moments that act wholly independently of each other on *OADB*, one along x and the other along y, just as truly as if they acted on two separate plates.

By symmetry $M_x=M_y$

By (2) and (3), we have

$$\tfrac{1}{2}dM_x=\tfrac{1}{8}(W-4wx^2)dx\ldots\ldots\ldots(4)$$

or

$$\tfrac{1}{2}M_x=\tfrac{1}{8}(Wx-4wx^3/3)$$

when taken between suitable limits.

Integrating (4) between $x=0$ (for a mere point of support) and $x=\frac{1}{2}L$ at mid span, we have for the quarter panel

$$\tfrac{1}{2}M_x=\frac{1}{16}(WL-\tfrac{1}{3}wL^3)=\frac{1}{24}WL\ldots\ldots\ldots(5)$$

Hence, the total increase of the bending moment between a median line of a panel and a parallel edge for the entire panel width is

$$M_x=\frac{1}{12}WL\ldots\ldots\ldots(6)$$

It can be shown in like manner that in case the panel is not square, but has a length L_x and a width L_y between column centers, that the total bending moments produced between the median lines and the edges are respectively

$$WL_x/12, \text{ and } WL_y/12.$$

The total bending moments are reduced below this in case of larger caps, by reason of their size and the consequent reduction of the effective span. For example, if we assume a cap having an effective length of edge of $C=\frac{1}{5}L$, which is as small as is ordinarily allowable, (4) must be integrated between the limits $x=L/10$ and $x=L/2$ and we find

$$M_x=WL/17+ \text{..........................} (7)$$

so that the total bending moment in the practical case is less than

$$M_x=WL/16$$

For a cap of still larger size the moment will be smaller, and in case of an edge of effective length $C=0.225L$, such as is frequently employed, we find the bending moment

$$M_x=WL/18+$$

It is to be particularly noted with respect to the foregoing demonstration that it is wholly independent of any consideration whatever of the shape or position of lines of zero shear, and is also wholly independent of any question of twisting moments. It rests solely upon the fundamental equations (3) of the necessary relations defined by all writers to exist between bending moment and shear, viz. that the increment dM_x of the bending moment between two right sections at a distance dx apart is the elementary couple that is the product of $\pm$ S_x the shear on those sections multiplied by their arm dx.

That this is the correct meaning of Eqs (3) has been called in question by those who have not recognized the fact that one part of the statical moment of the load consists of twisting moments; but a reference to Church's Mechanics, Art. 240, Merriman's Applied Mechanics, Art. 48, Morley's Strength of Materials Art. 59, will show that the definition given above is in strict accordance with accepted principles, so that bending moment and statical moment are not necessarily the same. In fact, it has been previously shown that they differ from each other wherever twisting or warping occurs. It is to be further noted that the bending moment here found is the same as that already obtained in Chapter III and confirms its correctness. It was there shown that the sum of the bending and twisting moments which are called into play in transmitting the load upon the panel to the supports is equal to the total statical moment of the load about the supports.

The numerical coefficient of *WL*, or the so called moment coefficient for caps of different sizes relative to the span is tabulated in the accompanying diagram, in which ordinates represent C/L the relative size of the cap, and abscissas express calculated values of the moment coefficient which diminish with the decrease of the clear span between caps.

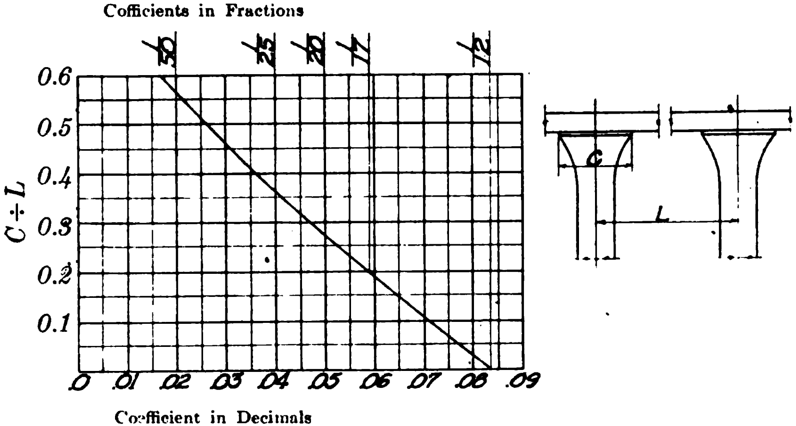


Fig. 122

Curve Showing Total Bending Moment Coefficient for Flat Slabs as Diminished by the Size of the Cap.

N. B. The effective dimeter of the cap is from 4 to 6 ins. less than the actual.

These coefficients apply only where there is such uniformity of distribution of metal radially and circularly, about the diagonal center of the panel and about the column centers respectively, as is necessary to make the slab imitate the action of a homogeneous plate. This uniformity is not departed from by providing a greater moment of inertia in the cantilever area corresponding to the greater moment there found than in the suspended span area, but it is departed from wherever the area of the mat is insufficient to cover the areas of synclastic curvature. Any considerable deviation from this relation so impairs the operation of the reinforcement that imitation of plate action is impossible, and the moment coefficients do not apply because of the different values of the twisting moments produced by the modification in the uniformity of the distribution of the metal.

Under the principle of rigidities, change in the resisting moment of inertia of the plate changes the distribution of moments of resistance. But where the same thickness of concrete is used throughout the slab and the flare of the capitals is depended on to give the proper shear areas under load, the only change that occurs in the moment of inertia is that brought about by distribution of the steel.

In the original Mushroom type, approximately 2½ times as much slab steel was distributed per square unit in a mat over the support as was distributed in the mat about the diagonal center at mid span. But the thickness of the mat over the supports reduced the lever arm of the steel in that part of the slab to such an extent that the total resistance normal to the column line at the top of the slab as compared with that normal to median lines of the panel parallel thereto would not vary greatly from that present in a plate of homogeneous material, and the rigidities over the support and between the supports for uniform load might be taken as corresponding approximately to that of homogeneous material.

This statement is true only where the reinforcing bars are of such size that they can be compactly placed vertically in the mat over the column head. Thus, four layers of 5/16″ or 3/8″ rounds in an 8″ slab over the column cap would be within the scope of the rule laid down, but four or five layers of 5/8″ slab rods in a slab of that thickness would not come within the rule. In the latter case the moment of inertia at the support would be reduced by so large a percentage that the subdivision of resisting moment would differ greatly from that in a homogeneous plate.

The effect of variations of the moments of inertia on the distribution of resisting moments will be considered later.

3. The Magnitude of the Total Bending and Twisting Moments and the Center Deflection in a Rectangular Panel of a Uniform Continuous Slab. In order to find the total bending moment between a mid section and a margin of a panel arising from a uniformly distributed load, in a continuous homogeneous flat plate supported on rows of columns, let O be the panel center and $OBDA$ be a quarter panel of a uniform homogeneous plate supporting a load of w units per square unit of area thruout, by means of the vertical shear across HJK one fourth of the perimeter of the column cap. Assume that in such a plate the loading is carried from every point by vertical shear directly toward the column center D until it reaches the column cap, since the uniform rigidity of the plate in all directions apparently requires that the shears be so distributed as to make the loading proceed to the supports along straight lines. Then the magnitude of the total vertical shear across any section such as PQ in Fig. 123 will be the product of w by the number of square units in the area

$OPQO_1$ regardless of whether the direction of the section PQ be perpendicular to OD or not. The intensity of this vertical shear will be larger the shorter the section is, provided it cuts off an equal loaded area. Hence a section in some other direction than PQ such for instance as a section perpendicular to OD will have a greater intensity of vertical shear than PQ but that fact is of no consequence in the following developments.

Take the axes of x and y along the sides OA and OB respectively and let $OA=a$, and $OB=b$. Let x and y be the coordinates of any point P upon the diagonal OD and $x+dx$, and $y+dy$ the coordinates of the consecutive point P'. Then the total vertical shear S_x across the section PF is the total loading on the area $OPFB$, hence $S_x=w(bx-\frac{1}{2}xy)$; and similarly the total vertical shear across PE is $S_y=w(ay-\frac{1}{2}xy)$, in which

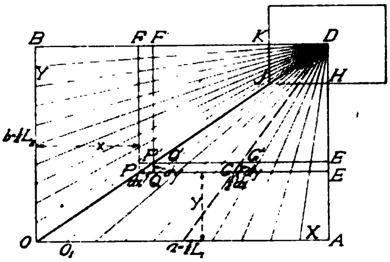


Fig. 123

$$a : x : : b : y, \text{ or } y=bx/a$$

Hence, $S_x=S_y$.

It is evident from the elementary triangles with vertices at D and bases each the same fraction of the median lines OA and OB into which the loaded area may be divided as shown in the accompanying Fig. 123 when the load proceeds in straight lines toward D, that the total vertical shear S_y is uniformly distributed along PE.

The center of action of S_y is consequently at G at mid point of PE and as P moves to P' the point G moves along the line GD to G',

Let M_x denote the total difference of bending moment between median line and margin acting about the axis of y and due to $2S_x$, which moment is distributed across the entire width of the panel, *i. e.* $\frac{1}{2}M_x$ = total difference of bending moments due to S_x between OB and the opposite margin of this area of one quarter panel.

Hence $\quad \frac{1}{2}dM_x=S_x dx=w(bx-\frac{1}{2}xy)dx$..............(8)

For convenience of computation assume that the length of the edges of the cap have the same ratio to each other as those of the panel.

Let the size of the cap be such that $DK = a/5$, then integrating between $x = 0$ and $x = 4a/5$

$$M_x = 2w\int\left(bx - \frac{bx^2}{2a}\right)dx = bw\left(x^2 - \frac{x^3}{3a}\right)_0^{4a/5} \text{ - - (9)}$$

Hence if $a = \frac{1}{2}L_1$, $M_x = 0.47a^2bw = 0.47\,\frac{WL_1}{8} = 0.94\frac{WL_1}{16}$ - - - (10)

Were we to take $DK = 0.225a$ the value of M_x would be still smaller.

If the support is a mere point then $DK = 0$, and in that case $M_x = \frac{2}{3}WL_1/8$. Similar values are likewise found for M_y.

Let N_x denote the total moment about the axis of y due to S_y for the entire width of the panel, *i. e.*, $\frac{1}{2}N_x$ = the total difference of moments about y due to S_y between OA and the opposite margin of this quarter panel, hence

$$\left.\begin{aligned} &\tfrac{1}{2}dN_x = \tfrac{1}{2}S_y dx = \tfrac{1}{2}w(ay - \tfrac{1}{2}xy)dx \\ \text{or}\qquad &dN_x = w(bx - \tfrac{1}{2}bx^2/a)dx \end{aligned}\right\} \text{-------------- (11)}$$

Hence $\qquad N_x = \frac{1}{2}M_x$, and $N_y = \frac{1}{2}M_y$

But the moment N_x is a twisting moment and not a bending moment at all, as it heretofore has been often tacitly assumed to be. From Fig. 123 it is evident from consideration of the vertical shears upon the faces *PQ* and *P′Q′* of an element that the only moment they have about y is due to displacement of the center of application on the face *P′Q′* to the right of that on the face *PQ*, which is merely an element of a twisting moment and not a bending moment, which total twisting moment between *PE* and *P′E′* has been calculated above about y correctly by multiplying the shear S_y upon *PE* and *P′E′* by $\frac{1}{2}dx$ the distance by which the center of action of S_y has been displaced from y in passing from *PE* to *P′E′*.

It therefore appears by this analysis that the only shears that operate to increase $\frac{1}{2}M_x$ are those designated above by S_x, and that the shears S_y that transmit the load resting upon the area *OAHJ* to the column and apply it as a vertical shear along *HJ* in doing so generate a total twisting moment $\frac{1}{2}N_x$ about the axis of y which is one half as large as the total bending moment $\frac{1}{2}M_x$ in a quarter panel between median section and margin. Now the resisting moment called into play in the slab to hold the applied bending moment M_x in equilibrium is subdivided into two parts, a resisting moment at the margin and one at median line, the relative size of these parts being controlled as is well known, by the relative rigidities of the panel at margin and mid span, but the distribution of each of these parts along the marginal and mid sections is not so well understood. This distribution of moment intensity depends in each case on two factors, viz., on the relative rigidity along the section and on the position of the point in the plate at which the bending moment to be resisted was originated by shear. It is evident from this that while the intensity of the resisting moment along the mid section of a uniform homogeneous plate will not be very much less near the panel center than near the edges, such is not the case along the marginal section in which the applied bending moment will be much larger per unit near the edges at the columns than between them.

Resistance to bending moments takes effect by producing longitudinal tensions and compressions at and near the surface of the plate in directions at right angles to the axis of bending, viz. M_x causes longitudinal tensions and compressions parallel to x, and M_y parallel to y.

But twisting moments are not resisted in this manner, for they are resisted by shearing resistances in parallel vertical planes, so distributed as to afford resisting moments equal and opposite to the applied twisting moments. But such shearing resistances call for no longitudinal tensions or compressions parallel to x or y. Hence the moments that resist bending are unaffected by the twisting moments arising from N_x and N_y.

Just as the vertical resistance at the section KJ holds S_x in equilibrium and supplies the upward force for the bending moments $\frac{1}{2}M_x$ about y in this quarter panel so likewise does the vertical resistance at the section HJ hold S_y in equilibrium and furnish the upward force to resist the twisting moment $\frac{1}{2}N_x$ acting about y in this quarter panel. It is an entire mistake to ascribe to this vertical supporting shear at HJ any additional function except that of redistributing the shears in sections like PE as the load between OA and HJ is carried to HJ. The shears in such distributed twisting moments as have been here discussed are readily resisted by a plate or slab in addition to other known stresses without producing dangerous conditions.

The twisting moment $\frac{1}{2}N_x$ about y which is produced by the redistribution of S_y, and the similar moment $\frac{1}{2}N_y$ about x produced by S_x is not to be confused in any way with the twisting moment independently produced by horizontal shears in vertical planes due to the kind of restraint to which the plate is subjected at the edges and to the consequent distribution of the longitudinal stresses arising from the bending moments. For example, in a continuous slab of many panels there is compression at and near the upper surface along each panel edge at mid span and tension across each edge which is replaced respectively by tension and compression at mid span at the lower surface. These states of stress are equivalent to simple horizontal shears on vertical plane sections which cut these edges obliquely but are of opposite signs above and below the neutral axis. Such opposite horizontal shears on oblique vertical planes constitute twisting moments about axes perpendicular to the oblique vertical planes mentioned. But these twisting moments due to horizontal shears are independent of those arising from vertical shears, as will be evident when we compare the case of a panel in a continuous plate such as has just been considered with that of a single isolated panel with free edges supported at the four corners either freely or built in. In this case of free edges there are longitudinal tensions and compressions along the edges as in the continuous plate but no tensions or compressions across the edge, and there are consequently no horizontal shears in vertical planes that cut the edges obliquely as in the continuous slab.

It is evident therefore that there are twisting moments due to horizontal shears in vertical planes that originate in the distribution of the longitudinal stresses arising from the distribution of M_x and M_y which distribution is controlled by the relative rigidities of the several parts of the plate and its supports, in their resistance to bending moments. These are entirely independent of the twisting moments arising from the distribution of the vertical shears which is controlled by the relative rigidities of the various parts of the plate in their resistance to vertical shears.

The general results reached by the foregoing detailed analysis may be established and made clear by elementary considerations, as follows:

The load upon the area *OAHJ* produces a total shear upon *HJ* equal to the load upon this area which has a limiting value of $\frac{1}{2}wab = W/8$. It also produces a total bending moment about x which is the product of the total load upon that area by its arm perpendicular to x, viz. the perpendicular distance of its center of gravity from *JH*. This arm approaches the limiting value of $2b/3 = L_2/3$ as the cap diminishes in size, and the load approaches $W/8$, so that the limiting largest value of the bending moment due to this load $\frac{1}{2}M_y = \frac{2}{3}\ WL_2/16$, for cap of vanishing size.

In addition to the shear and bending moment just mentioned the load upon this area in being transferred to the column also produces a twisting moment about y which is the product of the load by an arm perpendicular to y, which is the difference in distance from the axis of y between the center of gravity of the load and the center of action of the shear upon HJ. The limiting largest value of this twisting moment is $\frac{1}{2}N_x = \frac{1}{2}wab \times a/3 = \frac{1}{3}WL_1/16$.

The load upon the area $OBKJ$ acts in a precisely similar manner producing a shear on KJ of limiting value $W/8$, a limiting bending moment of $\frac{1}{2}M_x = \frac{2}{3}WL_1/16$ about y, and a twisting moment about x whose limiting value is $\frac{1}{3}\ WL_2/16$.

The distribution of the total twisting moment $\frac{1}{2}N_x$ along PE is governed by the amount of the shift of the several equal parts of the total vertical shear S_y along PE parallel to x as the load is transmitted from PE to $P'E'$ while an element of S_y situated at P is shifted a distance dx that at G is only shifted a distance $\frac{1}{2}dx$ while that at E suffers no shift whatever.

The intensity of the total twisting moment $\frac{1}{2}N_x$ which is distributed along PE increases uniformly therefore from E to its greatest value at P on the diagonal. Similarly the intensity of the total twist $\frac{1}{2}N_y$ along PF increases uniformly along PF from zero at F to its greatest value at P.

The results of this analysis are in full accord with those arrived at previously by two independent processes and may be regarded as a complete confirmation of the assumption here employed that the lines of zero shear are straight lines radiating from the supports, because it leads to results identical with those where no such assumption was made. This assumption therefore involves no contradiction of established theory.

4. Moments at Median Section and Margins of Interior Panels in Uniform Column Supported Flat Slab Floors When Uniformly Loaded. It has been shown in Section 2 that the total statical moment of the load $WL/8$, acting in a half panel between mid span and supports must be reduced by some factor dependent upon the effective reduction of span due to the size of the cap in order to obtain the sum total of the applied moments that must be resisted by the slab at mid section and at one side taken together, and further that this applied moment is subdivided in its action between a bending and a twisting moment. This statement may be put in algebraic form as follows

$$m\ WL/8 = M + N \text{..........................} (12)$$

in which m is the fractional multiplier by which the total statical moment of the load must be diminished on account of reduction of its effective span in order to obtain the actual applied moment $M + N$, of which the applied bending moment M is so related to the applied twisting moment N that $M = 2N$. Hence $\frac{3}{2}M = mWL/8$, or $M = \frac{2}{3}mWL/8$.

It has been further shown, that with the usually prescribed sizes of caps the value of the multiplier m is such that M is less than $WL/16$. Consequently the total applied bending moment to be resisted by the slab at mid section and edge of the cap when taken together is less than one half of the statical moment of the load between mid section and edge of the panel.

It has also been shown still further that a homongeneous plate or a floor slab that is initially flat and is reinforced in the tension zones by a fine-grained mat of crossed rods which is so firmly held together by the concrete matrix that it acts like a continuous sheet of metal when it undergoes synclastic curvature or double bending such as to produce nearly equal unit elongations e_s of the steel in two mutually rectangular directions in the dish shaped areas, requires the application of a bending moment in each of these two directions of more than twice the amount that must be applied in a single direction to produce an equal elongation e_s in that one direction alone, under cylindrical curvature. This is so because each of the moments that are mutually at right angles to each other not only produces positive bending in the direction of its action in its own plane but likewise at the same time produces negative bending in a plane at right angles thereto.

It therefore appears that in calculating steel deformations and slab deflections due to the total applied bending moments which act to produce longitudinal stresses across vertical planes mutually at right angles to each other the deflection will not exceed one half of the deflection of the same slab in cylindrical curvature as a beam under the action of one of these moments alone, but the total resisting moment shown by the longitudinal strains may be less even than one half that in the slab when considered as a beam under the action of the total applied bending moment in one direction only, because that total applied bending moment in case the slab is thick or has a thick drop may be resisted partly by arch action, and consequently only partly by the stresses called into play by the actual flexure of the slab. The moment acting to produce flexure alone is consequently the remainder of the total applied bending moment M

after it has been reduced by any arch action that may have been developed by the relative thickness, etc., of the slab or drop.

Since arch action in a slab makes use of part of the compressive resistance of the concrete to resist the horizontal thrust developed by deflections in the panel while the rest of it coacts with the tensions in the steel so that the elongations e_s of the steel at any point will be those due to the applied moments diminished by the amount of the moment that expresses the arch action, as will be explained later. Hence the computed stresses in the steel will be less than those produced by the total applied bending moment, but the panel deflections in case of continuous panels since they depend mostly on compressions in the concrete, will not differ materially from the deflections of flexure for the same total applied bending moment without such reduction. For purposes of design this total applied bending moment must be subdivided between mid span and panel edge in a ratio determined either by the ratio of the moments of inertia at these sections and by restraints at the supports or by the position of the points of inflection as investigated later.

By equations (8) and (9) the differential equation of the deflections z may be written in the form

$$EI\frac{d^2z}{dx^2} = \mathrm{m}_x = \mathrm{m}_o—wb(x^2\text{-}x^3/3a)\ldots\ldots\ldots(13)$$

in which the origin of x and z is at the center of the panel, m_x is the value of the applied bending moment acting at any vertical section of the panel situated at a distance x from the center, and m_o is that at mid section where $x=0$.

In this equation the moment of inertia according to Section 11 Chapter VI is.

$$I = \Sigma(Aijd^2)$$

It has a value that we may designate by I_0 at mid section of the panel, and another value I_x at the supports. These may be calculated as explained later in Section 6 by taking

$$A = A_1 + 2.5A_2$$

in which A_1 is that part of the cross section of the steel that resists by beam action, and A_2 the cross section of that part of it which is in synclastic curvature, assuming that the moment of inertia I_0 is constant from mid section to the points of inflection, and I_x is constant from inflection points to supports.

In case of uniform moment of inertia such as is now under consideration with $dz/dx = 0$, at $x = 0$, we have

$$EI\frac{dz}{dx} = \mathrm{m}_o x - wb(x^3 - x^4/4a)/3 \ldots\ldots\ldots\ldots (14)$$

If the slab is so restrained by the caps that it is horizontal at a point at a distance $x = nL_1$ from mid span, then at that point $dz/dx = 0$, and after introducing $x = nL_1$ and $W = 2wbL_1$, we obtain by (14) the moment at mid section.

$$\mathrm{m}_o = n^2(1-\tfrac{1}{2}n)WL_1/6 \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

and by (13) we obtain the corresponding moment at the margin where $x = nL_1$

$$\mathrm{m}_x = \mathrm{m}_o - n^2(1 - n/6)wbL_1^2 = -n^2(4 - 3n)WL_1/12 \ldots (16)$$

From these we find the total applied bending moment between mid section and margin is

$$M_x = \mathrm{m}_o - \mathrm{m}_x = n^2(3 - 2n)WL_1/6 \ldots\ldots\ldots (17)$$

The fractional coefficient n here introduced to express the distance of the edge of the cap from mid span is related to C/L previously employed to express the size of the cap as follows:

In any span L we have

$$nL = \tfrac{1}{2}(L - C), \text{ or } n = \tfrac{1}{2}(1 - C/L) \ldots\ldots\ldots\ldots (18)$$

so that we have corresponding values of these quantities and ratios of corresponding moments as shown in the following table:

*TABLE OF MOMENT RATIOS FOR GIVEN CAP RATIOS C/L

C/L	.0	.1	.2	.3	.4	.5
n	.5	.45	.4	.35	.3	.25
$\mathrm{m}_o/WL_1 = n^2(1-\frac{1}{2}n)/6$	.030	.026	.021	.017	.013	.009
$-\mathrm{m}_x/WL_1 = n^2(2-\frac{3}{2}n)/6$	.053	.045	.038	.030	.022	.016
$M_x/WL_1 = n^2(3-2n)/6$	.083	.071	.059	.047	.035	.025
$\mathrm{m}_o/M_x = \dfrac{2-n}{6-4n}$	.375	.37	.36	.36	.35	.35
$-\mathrm{m}_x/M_x = \dfrac{4-3n}{6-3n}$	.625	.63	.64	.64	.65	.65
$-\mathrm{m}_o/\mathrm{m}_x = \dfrac{2-n}{4-3n}$	.6	.58	.57	.56	.55	.54

*This table applies when I_x the moment of inertia over the Columns equals I_o the moment of inertia at mid span.

In this table m_o = applied bending moment at a panel mid section

m_x = applied bending moment at a panel margin

$M_x = m_o - m_x$ = the total bending moment applied between mid section and margin

The values of M_x/WL_1 in this table have been previously given in Section 2 of this Chapter. For example, in case the supports are mere points at the corners then $n = \frac{1}{2}$, and we have

$$m_o = WL_1/32, \quad m_x = -5WL_1/96, \quad \text{and } M_x = WL_1/12.$$

The results given in this table undergo modification in case the slab is not uniform by reason of variations of its moments of inertia where for instance the total moment of inertia at mid section differs from that at the margin. In cases, however, where these moments of inertia are equal it is evident from the table that the steel stresses at the margin will exceed those at mid section because $m_x > m_o$. On that account a designer would increase the marginal moment of inertia in order to make the steel stresses more nearly equal at mid span and margin and in so doing he would according to the principles of rigidities increase m_x at the expense of m_o and so decrease the ratio m_o/m_x to some extent. While the algebraic analysis of the case of unequal moments of inertia at medium section and margin meets difficulties of solution by reason of the degree of the unknown quantities involved, a satisfactory graphical analysis may be developed as follows:

5. Bending Moments and Moments of Inertia at Mid Section and Margin in Interior Panels of Continuous Column Supported Flat Slab Floors Uniformly Loaded. Let AO'' in Fig. 124 be the left hand half of a moment curve constructed as follows for a load on the span $L = 2AO$ between column centers where A is a column center, O is the mid span and the load is distributed in the same manner as it is on each of the diamond shaped areas of a slab on separated supports, viz. a load uniformly increasing from support to mid span which may be represented by the ordinates of the line AD. Subdivide the span L into some number of equal parts. In Fig. 124 $AO = \frac{1}{2}L$ is subdivided into 5 equal segments, then the ordinate of AD at the mid point of each of these segments may be taken to represent graphically the amount of that segment of the load. Lay these off successively on some convenient scale, downward from A, as the applied loads. On the same horizontal as the extremity H of the total load on the half panel assume P at a convenient pole distance from H, and connect it by a pencil of rays with the points of subdivision of the load line. Draw verticals thru the centers of gravity of the trapezoidal load areas. The moment curve may then be constructed as usual by drawing successive segments of it between these gravity verticals parallel to the successive rays of the pencil.

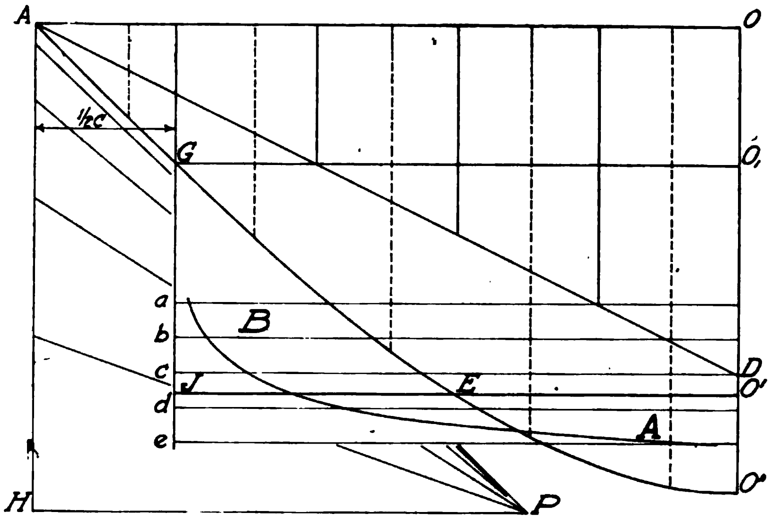


Fig. 124

The ordinates of this moment curve will represent the relative magnitudes of the bending moments produced in transmitting a load uniformly distributed upon a slab to its separated supports, because, as has been shown, these ordinates represent the only bending moments about the sides called into play by the total loading.

The side AE of this curve is straighter than the side of a parabola having an equal center ordinate OO'', and that part of the curve near the vertex O'' has a sharper curvature than such a parabola. This is due to the concentration of loading toward mid span at O. The two symmetrical halves of this moment curve do not constitute a single algebraic curve because the ordinates of the load curve does not follow a single law from column center to column center but undergoes a sudden change at O.

Let the effective span between column caps be $2GO_1 = L-C$, then GO'' is the left half of a moment curve due to that part of the load upon the effective span $L-C$, when C is the length of those portions of L occupied by the caps. In Fig. 124, $C = .2L$, and $L-C = .8L$. Assume some horizontal closing line of zero moments for this moment curve GO'' as JO'. We can assume JO' at such a level as to make the point of inflection E fall at any desired distance from mid span O'. If JO' is to be the left hand half of the actual closing line for an effective span $2GO_1 = 2JO'$ in which the center moment is $O'O''$ and the negative end moment GJ is to furnish the restraint required to keep the end horizontal, then the total bending in the span is zero. Now actual bending is proportional to total applied moment area and inversely as the stiffness or moment of inertia.

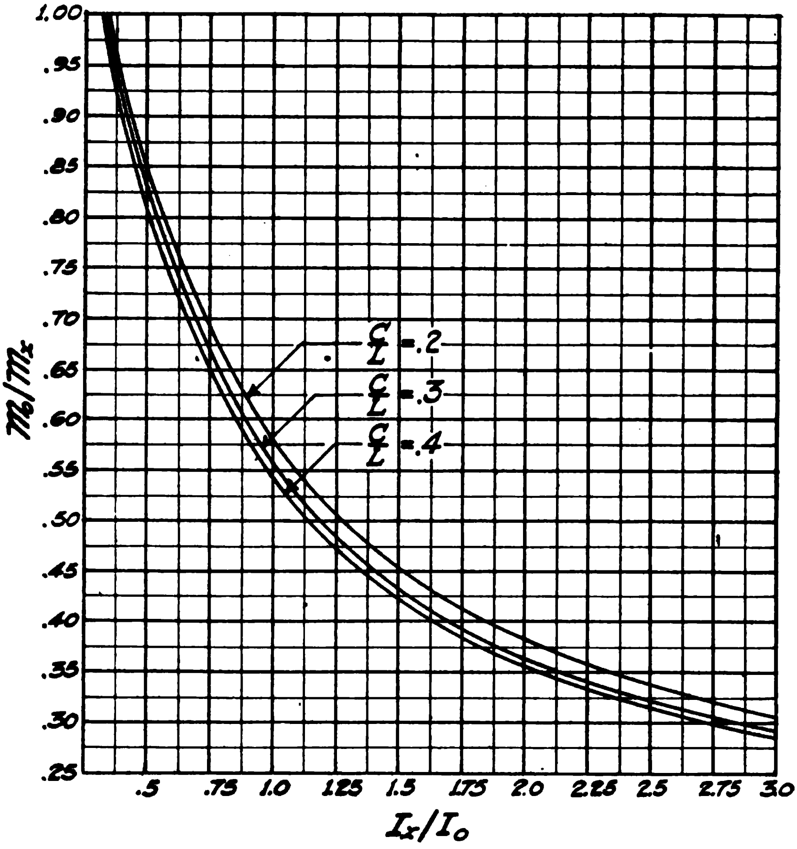


Fig. 125

Let I_0 be the moment of inertia of the central part of the span between points of inflection and I_x that of the end segments. Designate the area $EO'O''$ by A, and EJG by B. Then the condition that the slab remain horizontal at the ends and at mid span is expressed by the equation

$$A/I_0 = B/I_x, \text{ or } A/B = I_0/I_x \ldots\ldots\ldots\ldots (19)$$

Find the ratio A/B by planimeter measurements, and we thereby determine what ratio I_0 must have to I_x in order that the point of inflection will fall on the moment curve at the point where the assumed closing line JO' intersects it. In other words, if the slab be so designed that the values of I_0 and I_x have a ratio to each other of I_0/I_x equal to the measured ratio A/B, the point of inflection will be at the level of the closing line JO'. Now this required value of the ratio I_0/I_x can be secured by a suitable distribution and design of the reinforcement at the mid section and the margins of the panel.

Now assume several successive trial closing lines at nearby levels as at a, b, c, d, e for a given size of cap C/L, as for example for $C/L = .2$, and determine in each case by planimeter measurements the corresponding value of A/B. Then starting from the vertical G at the edge of the cap, lay off these values of A/B as abscissas on the horizontals thru a, b, c, d, e respectively to any convenient scale. Thru the points so found an interpolation curve may be drawn whose ordinates represent the levels of the closing lines a, b, c, d, e, etc., and whose abscissas express the corresponding values of $A/B = I_0/I_x$, and at the same time the levels a, b, c, d, e, etc., show the positions of the points of inflection where the bending moments vanish. By means of such an interpolation curve for a given value of C/L we may start with any given or assumed value of I_x/I_0 and find immediately the level of the closing line, the ratio $\mathrm{m}_0/\mathrm{m}_x$, and the position of the point of inflection.

The accompanying table has been obtained by this process. It covers such a range of values of C/L and I_x/I_0 as to include the usual sizes of caps and ratios of reinforcement at mid span and margin and shows moreover not only the mean position of the points of inflection that correspond to these but also the amount of the total moment in the effective span and its distribution between mid span and margin. By this table, therefore, when we have assumed any usual size of cap and any relative amounts of reinforcement at mid span and margin of the panel we are able at once to find the absolute and relative magnitudes of the bending moments at mid span and margin of the panel as well as the mean position of the points of inflection in the span.

MOMENTS AND POINTS OF INFLECTION IN DIRECT SPANS OF SQUARE PANELS

$\frac{C}{L}$	$\frac{I_x}{I_0}$	$\frac{2x_1}{L}$	$\frac{2x_1}{L-C}$	$\frac{m_0}{WL/2}$	$\frac{m_x^*}{WL/2}$	$\frac{m_0}{m_x}$	$\frac{m_0}{m_0+m_x}$	$\frac{m_x}{m_0+m_x}$
.2	.35	.540	.675	.0588	.0588	1.000	.500	.500
.2	.50	.516	.646	.0538	.0638	.844	.457	.543
.2	1.00	.455	.568	.0432	.0744	.580	.367	.633
.2	2.00	.385	.481	.0325	.0850	.383	.277	.723
.2	3.00	.350	.438	.0275	.0900	.306	.234	.766
.2	4.00	.328	.410	.0237	.0936	.253	.202	.798
.2	5.00	.305	.381	.0212	.0964	.221	.181	.819
.3	.35	.475	.677	.0462	.0475	.975	.494	.506
.3	.50	.450	.643	.0425	.0512	.829	.453	.547
.3	1.00	.396	.565	.0335	.0603	.556	.357	.643
.3	2.00	.335	.478	.0250	.0687	.364	.267	.733
.3	3.00	.305	.435	.0212	.0725	.293	.227	.773
.3	4.00	.280	.400	.0181	.0756	.240	.194	.806
.3	5.00	.270	.385	.0163	.0775	.210	.173	.827
.4	.35	.455	.675	.0350	.0362	.965	.492	.508
.4	.50	.385	.641	.0319	.0394	.809	.447	.553
.4	1.00	.340	.566	.0250	.0463	.540	.351	.649
.4	2.00	.290	.483	.0188	.0525	.357	.263	.737
.4	3.00	.265	.441	.0158	.0555	.284	.221	.779
.4	4.00	.245	.408	.0138	.0575	.237	.193	.807
.4	5.00	.230	.383	.0119	.0594	.200	.167	.833

In this table m_0 is the applied bending moment at mid section, and m_x that at the edge of the cap, I_0 the moment of inertia of the panel at mid section and I_0 that at the margin. In order to facilitate the use of the table the relation between I_x/I_0 and m_0/m_x is plotted in Fig. 125 which shows for the three values .2, .3 and .4 respectively of C/L how the ratio of the center moment m_0 to the marignal moment m_x diminishes as the relative stiffness at the margin I_x/I_0 increases. The striking character of the results shown in these curves makes it clear how little the ordinary assumption of distribution of the total applied bending moment into one third at mid section and two thirds at the supports corresponds to the actual distribution in reinforced slabs where I_x/I_0 differs from unity, as it usually does.

6. Moment of Inertia of Reinforced Flat Plates. The moment of inertia of a plate properly reinforced with a fine-grained mat when bent in synclastic curvature differs from the moment of inertia of a beam of the same span and reinforced with the same weight of metal because under synclastic bending a given weight of steel in a slab is somewhat more effective in each of two directions. This increase of effectiveness must be taken account of in comparing the resisting moment and relative moment of inertia of the reinforcement over the supports and at mid span between the supports, and in apportioning the applied moment which will be resisted at these critical sections.

In Chapter III Section 10, it was shown that where two equal stresses act on planes at right angles to each other in any continuous material the stress is the same on any other intermediate plane, so that a condition of plane hydraulic stress exists. Where the tensions in two belts of reinforcing rods that cross each other at right angles in a slab are practically equal they afford a resistance equivalent to that of a single sheet made up of the two belts and equally effective in all directions about their intersection and there is a certain additional increase in this effectiveness to be taken account of because of the double bending which is to be further taken into consideration under the principles demonstrated in Chapter VIII. That is, where the steel is in the form of small rods and closely spaced it is 2½ times as effective as though the same rods without change of position were operating under the same stress in cylindrical or anticlastic curvature. To acquire this full ratio of efficiency the area of the mat of crossed rods must cover the area of synclastic curvature. Otherwise, its efficiency is reduced in a ratio approximating the reduction of relative area for changes of 10 to 25 percent and at an increasingly rapid ratio for any reduction under these percentages which should be carefully avoided in practice.

Because the greatest moment to be resisted occurs over the supports it frequently becomes necessary to obtain this increased resistance over the supports by having the slab steel spliced over the columns, and the efficiency of the laps is added to the resistance at these points. The splice or lap therefore enters into the determination of the moment of inertia over the supports and if the lap is sufficiently long it may form an effective part of the mat. The length which renders the splice efficient is determined by the question whether the lap is long enough to extend well into the rim of the area of synclastic curvature where circumferential action is largely developed. Short laps serve only to reduce the measured stress along the length of the splice while adding relatively little in proportion to their cross section to the rigidity of the mat as a whole. Hence steel laps are to be considered as fully effective in increasing the rigidity and moments of inertia of areas of synclastic curvature only when they are long enough and at the same time are correlated to other equal laps crossing them.

Supposing the size of cap and the amount and arrangement of the reinforcement given in any design for a floor, the steel in the area of synclastic curvature around the panel center may be treated for the purpose of computation as 2½ times as efficient in every direction as a single belt in its own direction provided the reinforced area of synclastic curvature has a diameter of approximately 7/10ths of the panel width. The steel in the remaining 3/10ths of the panel width in anticlastic curvature has an efficiency the same as it would have in a beam section. This treatment is permissible only when the half diagonal of the mesh does not materially exceed the distance from the steel to the neutral plane, and the width of the crossed rod mat in the bottom of the slab is not less than 7/32nds of the sum of the sides of the panel, but 8/32nds gives a better distribution. Having thus determined the value of I_0 that of I_x is to be calculated in like manner.

Where the slab is of uniform thickness and a spider of large bars is used to impart shearing rigidity to the slab at the support the neutral plane becomes a scallop shaped surface instead of a plane surface around the supports. The moment of inertia added by the large radial rods is relatively small compared with the moment of inertia of the slab bars because the latter function as a mat which the larger rods cannot effectively do. The efficiency as a tensile flange element of the radial rods of the spider is therefore relatively small and an approximate estimate of their efficiency only is sufficient.

With the values of C/L and I_0/I_x thus ascertained for the design under consideration the table in section 5 will give m_0 and m_x. If the cross sections of the steel required by these bending moments at mid section and margin differ materially from those in the proposed design they should be adjusted to meet the requirements that they themselves involve.

The area of the steel section required at any section is to be found from the applied moment at that section from the equation—

$$\mathrm{m} = f_s I/id = f_s Ajd$$

as shown in Section 11 Chapter VI, in which A is the effective cross section of the steel acting as in a beam, but any steel that acts in synclastic curvature acts with a multiple effectiveness and can have its actual cross section diminished in a corresponding ratio to less than half that required in beam action. Hence the cross section

$$A = \mathrm{m}/(f_s jd)$$

is the cross section that would be required for beam action which may be written

$$A = A_1 + cA_2$$

in which A_1 is the area of that part of the steel section which acts in a beam, and A_2 the part in synclastic action which has an effectiveness c greater than 2. A_1 and A_2 need be taken only large enough to make $A = A_1 + cA_2$ satisfy the value of m given above.

The steel sections so obtained are however to be modified in turn to afford the necessary resistance whenever the size and moment of inertia of the columns cast integral with the slab is not sufficient to practically fix or restrain the slab to a horizontal position at the supports under the unbalanced loads to which it will be subjected.

At exterior panels, altho the loading may be taken as uniformly distributed the restraint at the outer edge of the floor measured by the moment of inertia of wall and wall columns is usually insufficient to hold the floor horizontal at the edge and considerations as to distribution of applied moments are to be introduced that are employed later for the case of unbalanced loads. It will be found that graphical constructions like that employed in this chapter may be used most advantageously, since they lead most directly to results of a high degree of accuracy.

7. Sources of Misunderstanding or Misconception Regarding the Operation of Flat Slab Floors. In considering a column supported flat plate, the structural engineer takes it for granted that the applied bending moment to which the material is subjected can be computed by ordinary beam theory, but this theory has been usually misapplied with the result that not only are the applied bending moments over-estimated as they should not be under beam theory, but the resistance of the material is also greatly under-estimated. Take the moments along diagonal lines about which the twisting moments may be disregarded since they are in equilibrium and figure these moments from the known shears that the loads produce across the diagonals by considering the center of gravity of the loads where they are known to be applied, then a rational application of beam theory to the problem will tend to make clear the fundamental errors that have been heretofore made in the determination of the magnitude of these moments.

Under uniform load, if the panel area between four posts be considered to be sub-divided by the median lines into four quarters, it is apparent that the load on each quarter is tranferred from its center of gravity at the quarter point of each diagonal to the adjacent column, and if the column support be considered to be merely a point, the total applied moment produced by the transfer is $W/4 \times L_d/4$ in which L_d is the diagonal span, or $L_d = L\sqrt{2}$ where L = the direct span if the panel is square, or in any case L_d = the square root of the sum of the squares of the sides of the panel whether the panel be rectangular or square.

In transferring the load to the four columns the material of the panel resists in two directions diagonally and the distribution of the quarter panel load along the diagonal which goes to each of the diagonally opposite columns is one that increases uniformly from zero at the column to a maximum at the quarter point of the diagonal and from that point decreases uniformly to zero at the diagonal center of the panel, and again increases on the other side of the center so that the moment curve of the load so distributed along the diagonal differs from the moment curve of a uniform load in that it is much steeper at the supports and much flatter throughout its center portion than the parabolic curve of uniform loading.

The effect of this distribution of load is to move the point of inflection on the diagonal toward the support as compared with its position in a continuous uniform beam under uniform load. Furthermore, it makes a cap of given diameter have an effect to reduce the total applied moment much more than would be the case were the load uniformly distributed. By reason of the fact that the load is divided in its diagonal travel in two directions the total load transferred by each diagonal is only half the total panel load, and the loads which travel in the two directions are separated by the median lines of the panel which are known axes of zero shear. The sum of the bending moments along the two diagonals considering the supports to be mere points is thus $WL_d/8$ as required by beam theory, or $WL_d/16$ in each direction. One correct application of beam theory therefore takes the diagonals of the panel as the spans for two beams that cross each other with two loads each equal to $W/4$ on each beam, the centers of gravity of which loads are situated at the quarter points of the diagonals. These four loads comprise the entire panel loading, and since they are each separately transferred to the nearest support the only bending moments they can produce on the diagonals are those just calculated.

A different application of beam theory also correct has been made previously where the entire load upon the floor was separated into lozenge shaped areas by the diagonals. These lozenge shapes areas each extended along a side of a panel, and the load upon the area produced bending moments in transmitting the load to the supports at the ends of its sides, and these were the only bending moments required to transmit the loading to the supports. The twisting moments about the axis of each lozenge are in equilibrium and need not be considered in connection with the bending moments.

These two separate analyses of bending moments along diagonals and along the sides are in no sense contradictory. They are independent and concordant. The position of the points of inflection on the panel sides for several sizes of caps and ratios of moments of inertia, etc., at mid span and supports, have been already determined graphically. The position of the points of inflection on the diagonals, etc., and the apportionment of the total bending moment between mid span and supports on the diagonals are arrived at by a precisely similar graphical process on the diagonals each under two loads of $\frac{1}{4}W'$, as that previously described, while the determination of the moments of inertia at mid span and supports requires special consideration.

The moments of inertia to be employed in apportioning the total moment between the moment over the support and that at mid span will be, first I_x the moment of inertia over the column head figured as before, and second I_0 the moment of inertia at mid span which will be the sum of the moments of inertia across the panel on median lines in each direction.

This is evident since the metal about the diagonal center operates as a sheet and a sheet resists stresses in two directions at right angles to each other with even greater efficiency than it can resist stress in one direction. Thus the crossed rods in the mat about the diagonal center of the panel are able to perform more than double duty, if sufficiently widely distributed, while the steel in the areas of anticlastic curvature which extends directly to adjacent columns, supports none of the load on the diagonal strip considered, and consequently it acts with single efficiency on both sections just as it would in the lozenge shaped strips directly between columns.

Because the double moment of inertia at the diagonal mid span opposes the bending of the segments of the adjoining quarter panels, the point of inflection on the diagonal is pushed toward the column to a percentage of the total length of the span which is usually not far from .8 of the percentage of the span length at which the point of inflection lies on direct column lines when the direct and diagonal span is taken as center to center of columns in the two respective directions.

MOMENTS AND POINTS OF INFLECTION IN DIAGONAL SPANS OF SQUARE PANELS.

$\frac{I_x}{I_o}$	$\frac{C}{L}$	$\frac{2x_1}{L}$	$\frac{2x_1}{L-C}$	$\frac{m_x}{\frac{1}{2}WL}$	$\frac{m_x}{\frac{1}{2}WL}$	$\frac{m_o}{m_x}$	$\frac{m_o}{m_o+m_x}$	$\frac{m_x}{m_o+m_x}$
.5	.15	.580	.682	.0325	.0057	.583	.368	.632
.5	.20	.547	.685	.0275	.0492	.558	.358	.642
.5	.25	.515	.687	.0242	.0424	.548	.353	.647
.5	.30	.484	.692	.0190	.0355	.535	.349	.651
1.0	.15	.524	.617	.0243	.0630	.379	.275	.725
1.0	.20	.442	.615	.0200	.0568	.353	.261	.739
1.0	.25	.467	.622	.0170	.0485	.350	.259	.731
1.0	.30	.435	.626	.0140	.0405	.345	.257	.743
2.0	.15	.468	.551	.0170	.0712	.239	.194	.806
2.0	.20	.442	.552	.0145	.0623	.233	.189	.811
2.0	.25	.413	.550	.0120	.0535	.224	.183	.817
2.0	.30	.390	.557	.0100	.0445	.225	.184	.816
3.0	.15	.430	.506	.035	.0747	.181	.154	.846
3.0	.20	.407	.508	.0115	.0653	.176	.149	.851
3.0	.25	.386	.514	.0095	.0560	.170	.145	.855
3.0	.30	.365	.521	.0083	.0463	.178	.152	.848

In case of a rectangular panel the load distributed along the diagonal is constant for some distance each side of each quarter point and then decreases uniformly to zero at the diagonal center and at the column center.

The total applied bending moment in the diagonal span obtained by the preceding process when due regard is had to the practicable diameters of column caps is found to give the same total applied bending moment as that obtained in the lozenge shaped side spans in a square panel, and is approximately equal to a moment similarly figured in a rectangular panel when the mean span is taken as $\frac{a^2+b^2}{a+b}$, a and b being the long and short sides of the panel respectively. This determination is in accordance with the principles of plane hydraulic stress demonstrated in Chapter III, Section 10. The following table gives the coefficients locating the point of inflection, and the apportionment of the moments at mid span and over the supports for a rectangular panel in which L_1, the long side, equals 1.2 times L_2 the short side.

MOMENTS AND POINTS OF INFLECTION IN DIAGONAL SPANS OF RECTANGULAR PANELS

For $L_1/L_2 = 1.20$

$\frac{I_x}{I_o}$	$\frac{C}{L}$	$\frac{2x_1}{L}$	$\frac{2x_1}{L-C}$	$\frac{m_o}{\frac{1}{8}WL}$	$\frac{m_x}{\frac{1}{8}WL}$	$\frac{m_o}{m_x}$	$\frac{m_o}{m_o+m_x}$	$\frac{m_x}{m_o+m_x}$
.5	.15	.577	.678	.0327	.0551	.592	.372	.628
.5	.20	.553	.692	.0288	.0475	.606	.378	.622
.5	.25	.519	.692	.0238	.0414	.576	.366	.634
1.0	.15	.523	.617	.0243	.0635	.383	.277	.723
1.0	.20	.500	.625	.0214	.0548	.389	.280	.720
1.0	.25	.468	.624	.0176	.0477	.368	.269	.731
1.5	.15	.488	.574	.0198	.0679	.291	.226	.774
1.5	.20	.466	.582	.0171	.0592	.289	.224	.783
1.5	.25	.437	.583	.0142	.0511	.277	.217	.776
2.0	.15	.458	.538	.0173	.0704	.246	.198	.802
2.0	.20	.439	.550	.0144	.0618	.233	.189	.811
2.0	.25	.413	.551	.0124	.0528	.234	.190	.812
2.5	.15	.447	.525	.0153	.0724	.211	.174	.826
2.5	.20	.423	.528	.0128	.0635	.202	.168	.828
2.5	.25	.402	.535	.0113	.0540	.208	.172	.833

L in the table is the length of the diagonal, $2x_1$ is the distance between points of inflection on the diagonal, $L-C$ is the distance along the diagonal between caps, m_o and m_x are the bending moments at mid span and supports, and I_o and I_x are the corresponding moments of inertia.

In the paper by Mr. Trelease, Proc. Nat'l Assoc. Cement Users, 1912, p. 229, Fig. 11, the points of inflection on the diagonals of a rubber model of a flat slab floor are shown nearly twice as far from the columns as they are upon the sides. But these results based upon tests of a rubber sheet failed to distinguish the predominant action in the sheet due to stretching or inverted arch action from the phenomenon of pure flexure with the surprising result that the points of inflection on the diagonals were located two and half times as far from the columns in proportion to inflection points on the sides of the panel as their true position is under pure flexure. Since a small variation in the position of the points of inflection makes a wide variation in the relative magnitudes of the moments over and between the supports, the grave consequences of any such misconception are apparent when applied to the design of a floor.

Under-estimation of the resistance of the material in synclastic curvature has led to defective designs in many cases in which no provision at all has been made for so restraining the synclastic cantilever areas to a synclastic mode of operation that any material transition from synclastic to cylindrical curvature is prevented under unbalanced load. Accordingly not a few designs that make a good showing under uniform load are lacking in strength under unbalanced loads. In the pioneer type, a heavy spider was utilized to furnish the required restraint. In the more advanced Spiral Mushroom Type diagonal strips and spirals combined with a lighter spider are the elements which operate to this end.

8. Deflections in Panel of a Continuous Column Supported Flat Slab Floor Uniformly Loaded.

Integrating equation (14) Section 4 we find the deflection z_1 between $x=0$ and the point of inflection $x=x_1$.

$$EI_o z_1 = \tfrac{1}{2}(\mathrm{m}_o x_1^2 - wbx_1^4(1-x_1/5a)/6) \ldots\ldots\ldots (21)$$

in which z_1 = the magnitude of the deflection between the origin 0 at the panel center Fig. 123 and that at the point of inflection on the diagonal at a distance $x=x_1$, and I_o the moment of inertia is assumed to be the same at all sections between $x=0$ and $x=x_1$, as it is at $x=0$. In (21) the value of x_1 is to be taken from the table in section 5 as determined by the known value of n and I_x/I_o.

Again by equation (14) integrate between the point of inflection at $x=x_1$ and the edge of the cap at $x=nL$, then

$$EI_x(z_2-z_1) = \tfrac{1}{2}(\mathrm{m}_o L^2 n^2 - wbL^4 n^4\ (1-\tfrac{1}{5}n)/6)$$
$$-\tfrac{1}{2}(\mathrm{m}_o x_1^2 - wbx_1{}^4(1-x_1/5a)/6) \ldots\ldots\ldots\ldots (22)$$

Hence $$EI_x z_2 - E(I_x-I_o)z_1 = \tfrac{1}{2}(\mathrm{m}_o L^2 n^2 - wbL^4 n^4(1-\tfrac{1}{5}n)/6) \ldots . (23)$$

from which z_2 can be obtained when z_1 is known from (21).
The total central deflection is

$$D = z_1 + z_2$$

9. Arrangement of Reinforcement in the Spiral Mushroom System. The original four way reinforcement, Figs. 5 and 6, concentrates the steel in the cantilever area over the columns where it can act as a mat. The disadvantage of the arrangement consists in having four layers of reinforcement one above another in which the center of action of the metal is removed to a considerable distance from the extreme fibre or top surface of the slab. Unless standard specifications were strictly observed, which require the splicing of belts in their own plane, it sometimes happened that instead of having only four layers of rods over the supports, eight layers were cob-piled there, and the center of resistance of the steel was then removed to a correspondingly large distance from the surface with a resulting tendency to over-strain and crack the concrete in tension in addition to reducing the efficiency of the material as a tensile reinforcement by emplacement too far from the outer fibers, which excessive cob piling, usually resulted in increased thickness of the slab in order to cover the steel. The elevation of the head being determined by the bend of the column bars and the length of the column reinforcement which extends down to the lower floor, the strength is thus preserved but the contractor in order to save what material he could in the finish in order to make up for this loss in the amount of concrete in the rough slab would frequently make the finish so thin over the support that it would become cracked by shrinkage and present a bad surface. These difficulties, together with the consideration that the concentration of the metal in the cantilever area causes a like concentration of the compressions about the column head led to the improved Spiral Mushroom, illustrated in Fig. 126, in which the various stages of placing the steel are shown.

Stage one shows the frame or spider on which the flat spirals are placed, the spider keeping the spirals at the proper elevation near the top of the slab. In the second stage, series of rods of a width approximately two thirds the diameter of the spiral are placed in two directions parallel to the column lines, these rods being at the top of the slab over the spirals and dipping sharply downward to the bottom of the slab beyond the line of inflection. Then diagonal strips from mid span to mid span of the sides are placed in the bottom throughout as shown in the third stage. And finally in the fourth stage it is shown how parallel rods in two directions fill in the central area, each set of rods rising to the top of the slab over the saddle back areas and dropping to the bottom therefrom and continuing through the bottom of the slab in the central half of the plate.

It thus appears that the entire area of the slab is completely covered by parallel rods in two directions which arrangement is effected in four stages as just described in order to be able to place these rods at proper levels with respect to the rest of the reinforcement in a convenient manner.

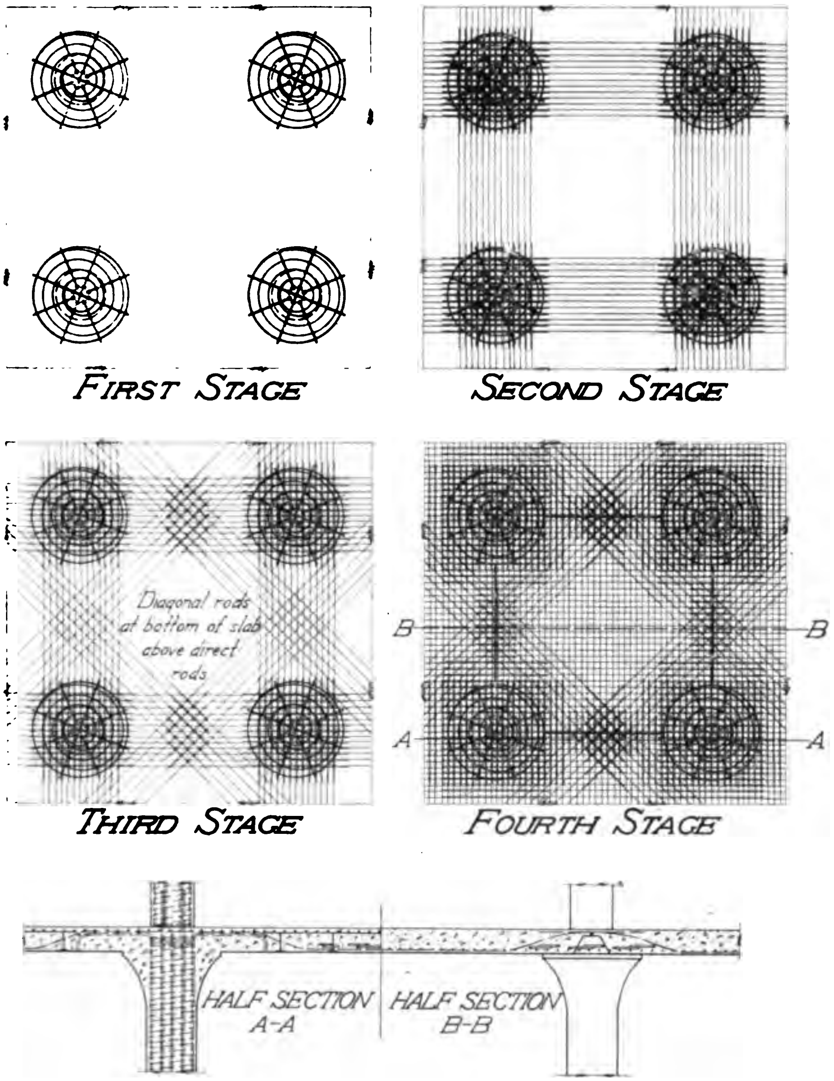


Fig. 126

The advantage of the Spiral Mushroom over the original form of construction, lies first in the reduction of the thickness of the mat of slab steel over the supports. The thickness of the mat of slab metal is reduced over supports to two layers of slab rods and a spiral rod which is generally about the size of the slab rods. There are accordingly only three thicknesses of steel in place of five layers originally which consisted of four layers of slab rods and the larger rings when the slab steel is spliced in the plane of the belts. The distribution of metal all along the saddle back areas normal to the column lines reduces the compression at the column capital, relieves the concrete of horizontal shears in vertical planes required to distribute the bending moment at the saddle back in the older type and adds considerably to the stiffness and greatly to the toughness of the slab as a continuous plate.

Comparing the distribution of metal in the Spiral Mushroom with the original, the areas of one layer of metal at mid span on the column lines in the original are eliminated by reinforcement across the panel sides. This is particularly advantageous in wall panels where a beam runs along the side of the panel, for in that position in the older construction the greater cross section of metal in two layers at mid span of the diagonal favors cylindrical curvature in a manner avoided by the localization of the maximum amount of metal at mid span across column lines in the new construction. So much synclastic curvature about the diagonal center of wall panels as is essential to imitation of plate action under unbalanced loading is thus assured.

The advantage secured by reducing the thickness of the mat over supports in the new construction is in a measure off-set by the fact that the reinforcing metal between the cantilever areas normal to the saddle backs is efficient in but one direction instead of operating solely on the principle of the mat as in the older type. On the other hand, the considerable vertical distance apart of the bars in the older type tended to interfere with the perfect operation of the metal as a mat. So that the balance of advantages and disadvantages produce the net result that checking or cracking of the slab because of possible erroneous cob piling is eliminated and much greater toughness and ability to withstand over-load is secured in the later improved type brought about in part by the distribution of compressive stresses more uniformly along the bottom edges of the panels and the reduction of twisting shears distributing moments along the sides.

The oblique two way reinforcement of the Spiral Mushroom floor slabs crossing in directions parallel to the diagonals of the panels consists of belts of rods in the bottom of the slab extending from mid span to mid span of the panel sides as shown in Fig. 126, where they together with the side belts form a mat of considerable stiffness and rigidity which not only assists in bearing the tensile stresses at mid span of the sides, but by increasing the stiffness of the saddle back areas in all directions affords resistance more nearly like a unitary structure with a self-contained system of stresses, less subject to warping and bending unsymmetrically in either direction than one with the same amount of steel all placed in parallel relation. This arrangement tends to equalize the stiffness of the saddle back with the circumferential cantilever and so to tie them both together as to increase their efficiency by their coaction with each other.

10. Computation. In considering the cross section of metal, that over the cantilever may be treated as a mat along column lines. That in the top of the slab between the cantilever edges is to be treated as developing only the equivalent of its cross sectional area in resisting the applied moment on the beam principle since there is no circumferential action possible in the saddle shaped portions of the slab.

On the cross section at mid span between supports, seven tenths of the width of the panel may be considered as within the dish shaped area and the cross section of one layer of rods therein taken as equivalent to two and one half times its cross sectional area in beam action and the cross section of the rest of the reinforcement is to be added thereto in making up the necessary resisting section of steel.

The oblique rods, while they have an amount of mat action that makes them about twenty five percent more efficient than would be due to their sectional area when in parallel, they are usually treated as equal thereto. No further allowance is necessary for partial loading when the column diameter is not less than 1/12th of the span and when the working load is not greater than 250 pounds per square foot, than to take the steel area in the dish shaped portions at 80% of its efficiency under uniform loads. Other cases are to be investigated by considering the rigidity of the column and the change in inclination at the support, as will be explained in a later chapter.

With the Spiral Mushroom type, the lines of zero shear differ but slightly in their position from those in the older type. The foci of these lines are removed from the corner of the panel outward a very short distance along the diagonal lines by reason of rigidity added to the sides by the oblique belts.

11. Flat Slab and Column Construction With Increased Rigidity in the Cantilever Portion and Saddle Back Areas. For very light loads and long spans, a flat plate of uniform thickness must either be made excessively thick or in some way an increase in rigidity must be made over and above that which would attend the design of such plates for usual working stresses since under such stresses excessive deflection might result. This is readily effected by increasing the moment of inertia in the cantilever and saddle back areas, using in the saddle back areas a wide drop or increase in thickness amounting to $2\frac{1}{3}''$ to $4''$ and a quarter of the panel length in width extending from column. Such a change increases the rigidity of the cantilever without lessening any of the advantages of plate action therein. It increases the stiffness of the saddle shaped area which operates between the cantilever ends on the beam principle in any case, and permits a thin light plate to be supported between the saddle shaped areas.

The lines of zero shear in the suspended plate are thus moved on the diagonal outward, away from the center of the column to some extent and the plate itself can be figured approximately as a circular plate 1-1/8th times the side of the rectangle in diameter. The panel is to be treated along column lines in the same manner as in the preceding cases, but the suspended span between the cantilever ends and the saddle back area may be treated on the beam principle by designing it to carry the load on the saddle back area plus approximately one sixth of the load on the suspended span.

12. Steel Ratios and Minimum Thickness of Slab. Since slabs should be so designed that the steel would yield before the concrete, it is important to determine how large a percentage of steel may be introduced without passing this limit. This, however, is dependent upon the relative thickness of the slab. In case of the continuous slab, the ratio of the depth to side span involves somewhat different considerations from that of the ordinary beam since the minimum thickness of the slab on a diagonal span is relatively less than is permissible in beam construction. That part of the thickness of the slab which serves as fireproofing and which is constant for any given size of rods bears a greater ratio to the depth than is the case in beam construction. Furthermore, the depth necessary for proper embedment over the support and the massing of the steel at the support reduces the effective lever arm of the steel in the cantilever to j_2d, which reduces it to a relatively greater extent than at mid span where the effective depth becomes j_1d as it does in the case of one way slab construction also. This reduction in effective depth for fireproofing, embedment and massing of the steel is, however, a constant for a given size and arrangement of reinforcement and for the continuous slab it is found that we may accordingly take this constant as follows:

2 " for 3/8" rod reinforcement (or smaller sizes)
2¼" " 7/16" " "
2½" " 1/2" " "
3 " " 7/8" " "

The limiting minimum thickness of slab then may be stated as $L/48$ plus the constant just tabulated for the various sizes of rods.

In case square twisted rods are used, the same constant should be used for 3/8" square twisted as for 1/2" rounds, and for 1/2" square twisted the same constant as for 5/8" rounds. This will give a satisfactory minimum thickness of slab for all spans of this type, L being the longer direct distance between column centers.

For square panels supported on beams or walls at the edges, the constant for the various size of reinforcing rods should be one inch less than that tabulated above, and the minimum thickness taken as $L/48$ plus this new constant.

Where the panel is rectangular and not square, and supported on the sides by beams, the same rules will hold by using $(a^2+b^2)/(a+b)$ in place of L for the span of the square panel, in which a and b are the spans in the two directions respectively.

Determination of reinforcement required in slabs. In case of beams with different values of N the ratio of depth to span, the proper values of the steel ratio p were obtained in Chapter VI page 166. In case of slabs the ratios so obtained should not be exceeded, but in reckoning the reinforcement of slabs account must be taken of the fact that the steel is in multiple directions so that the reinforcement per belt will be only a fraction of the total permissible steel.

In the continuous slab with four way reinforcement and Mushroom heads, having given the ratio of the thickness to span taken center to center of columns, determine the percentage of steel from the diagram for beams, page 166, and divide this by 2 to find the limiting percentage of steel for each belt.

In the slab with two way reinforcement, supported on columns, the side belt may be made a little heavier than in the four way belt type, .6 of the percentage given for beams being permissible.

In slabs not reinforced with a supplemental cantilever frame, but with a depressed head instead, the minimum thickness should not be less than the minimum thickness allowed for the former type, but the steel ratio may be made .55 that for beams instead of .5 which is permissible with mushroom heads.

In square slabs supported on beams or walls the percentage of steel in the strip occupying the middle third of the panel each way should not exceed that for beams of similar proportions of depth to span. In case the panel is a rectangular panel, the percentage of steel should not exceed that for square panel having a side equal to $(a^2 + b^2)/(a + b)$ which we have previously used in determining the minimum thickness of span for this form of panel.

With the above interdependent limitations as to the minimum thickness and maximum steel ratio in mind the designer has to determine how far in any given design he shall deviate for any reason from them. By making N smaller and the slab consequently thicker he will be able to reduce the deflection to such figures as may be desired or required. But when such deviations are made the proper relationship between steel ratio and slab thickness is still to be determined from the diagram page 166 as has just been done for the case of minimum thickness.

The erroneous requirement is made in some building codes that the maximum deflection of a slab under test shall be a no greater percentage of length of span than that permitted or allowed for a deep beam. A deep beam might be broken and seriously injured under a deflection which would do no damage whatever to a long thin slab. A certain degree of stiffness, however, is required under working loads wherever there are partitions in a building which may be damaged by undue deflections. The deflection in a thin slab should not exceed 1/700th of the span under working load, and preferably less. In an office building or an apartment house, where there are partitions which might be cracked by deflection, the deflection should be limited for working loads to 1/1000th of the span.

The inappropriateness of any requirement limiting the deflection to a given fraction of the span will appear from the following investigation of the elementary relations between the deflections and stresses in the steel of reinforced beams, since similar considerations apply to slabs. Using the notation previously employed we obtain from the well known expression for the steel stress f_s, the equation

$$W\,L\,(1 - k)\,d = n\,I\,f_s,$$

in which $n = 4, 8, 8$ or 12 for four different cases, namely simple and restrained beams carrying a load W either concentrated at mid span or uniformly distributed.

Again the well known expression for the deflection D gives us the equation.

$$W\,L^3 = m\,E\,I\,D$$

in which $m=48$, 76.8, 192 or 384 for the four cases mentioned

By combining these equations to eliminate W the following relation between f_s and D is obtained:

$$D = \frac{n\,L^3\,f_s}{m\,(1-k)\,d\,E}$$

This shows that the maximum deflection D in beams of different depths d, and the same length and unit steel stress f_s varies as $1/d$ while the coefficient n and m make further wide variations, which make it absurd to limit the permissible deflection to any given fraction of the span.

For test loads, however, greater deflections are permissible than for the working loads discussed previously. Actual test under common conditions of partial restraint shows that in a span of forty times the thickness or depth of the slab *i. e.*, $L=40\ d$, a deflection equal to $L/250$ will not materially injure the construction. Consequently in case $L=10d$, a deflection of $L/1000$ should not be exceeded.

Since a practical constructor desires to test only to safe limits rather than to anything approaching the ultimate limit he will not object to regulations twenty percent more rigid than those just mentioned provided the time of making the test is not less than four months after casting the concrete.

13. Increase in Depth of Slab Required for Increase of Load. For a given span and given size of capital the moment to be resisted increases directly as the load. Now were the area of the steel increased in the same ratio as the slab thickness the resistance of the slab would be increased as the square of the depth. Therefore, from this consideration the thickness would increase in the ratio of the $\sqrt{w}$ in which w would represent the sum of the dead and live loads per unit area. On the other hand, as the thickness of the slab increases relative to a given span the percentage of steel that it is permissible and economical to use also increases as before shown. Moreover, the amount of arch action increases with the thickness of the slab relative to a given length of span, while the dead weight per square foot as well as the thickness, increases at a less rapid rate when the percentage of steel is increased, so that the sum total o these variable elements renders the increase in net thickness or depth of the slab, from the center of the steel to the center compression fiber, nearly as the cube root of w, a conclusion which is in accord with good practice and quite different from the Joint Committe rule that the depth from the center of the steel to the top of the slab should increase as the square root of w, the loading. For the same load per foot, as the span increases, the bending moment for a given ratio of the diameter of cap to the span length increases as the cube of the span, and since the resistance of the concrete and metal for a given percentage of steel increases as the square of the depth, the increase in thickness of the slab with the increase in span for the same unit load should be directly as the increase in span. The fireproofing should be the same for each.

Eddy, Trans. Am. Soc. C. E. Vol. LXXXI pp. 1173, gives the following formula and Table for d the distance from the center of the steel to the the top surface at mid span:

$$d = L\frac{4w_0 + 2{,}800}{10^4} \text{ inches; = effective thickness}$$

L = span, in feet; w_0 = live load, in pounds per square foot.

Table 3.—Effective Thickness d of Flat Slabs.

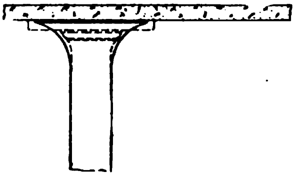
Span in feet	$w_0 = 100$	200	300	400	500	600	Differences
14	4.48	5.04	5.60	6.16	6.72	7.28	0.56
16	5.12	5.76	6.40	7.04	7.68	8.32	0.64
18	5.76	6.48	7.20	7.92	8.64	9.36	0.72
20	6.40	7.20	8.00	8.80	9.60	10.40	0.80
22	7.04	7.92	8.80	9.68	10.56	11.44	0.88
24	7.68	8.64	9.60	10.56	11.52	12.48	0.96
26	8.32	9.36	10.40	11.44	12.48	13.52	1.04
28	8.96	10.08	11.20	12.32	13.44	14.56	1.12
30	9.60	10.80	12.00	13.20	14.40	15.60	1.20
32	10.24	11.52	12.80	14.08	15.36	16.64	1.28
Differences	0.64	0.72	0.80	0.88	0.96	1.04	

This formula gives satisfactory values of d for the cap diameters of .2 the span length but requires modification and adjustment for other ratios of cap to span length which may be readily determined from the relative magnitude of the coefficients of the applied moments. It will be noted that these thicknesses follow very closely the above approximate relations, and to these thicknesses is to be added the necessary fireproofing.

14. Form of Capital and its Effect on Bending and Twisting Moments. For rectangular panels such as shown in Fig. 123, the analysis of the bending and twisting moments at the edge of the column cap is applicable to the edge of a depressed head also and the moments in either direction may be figured in the manner therein given, if the sides of the head are proportional to the sides of the panel. Where the column capital is round or octagonal the diagonal axes of zero shear may be treated as a straight lines just as with the rectangular form shown in Fig. 123. Where, however, the form of the panel is rectangular but a deep square plinth or drop head is used, this treatment is no longer exact because the diagonal axis of zero shear *OJD* of Fig. 123 is no longer a straight line but a broken line *OJ* to the corner of the square drop and *JD* at 45° from the corner of the drop to the column center. This change in the direction of the diagonal axis of zero shear causes a change in the amount of the twisting moments, which in the short direction equals approximately the difference of the coefficients in the two respective directions found in the curve of Fig. 122 and this proportionate part of the bending moment in the long direction must be added to the bending moment in the short direction to provide for this decrease in twisting moment normal to the long side.

The same result may be arrived at geometrically by treating the areas by the simple method given in Chapter III Section 23. In the long direction because the triangular area which produces this change of twist has its center of gravity one third of the distance from mid span to the edge of the capital removed therefrom in the long direction instead of two thirds as in the short direction the proportionate difference in the coefficients is to be divided by two as the ratio of the bending moment in the short direction determined from the coefficients of Fig. 122 to be added to the bending moment in the long direction of the panel.

15. Economic Form of Column Capital. In Fig. 127 a pattern of the capital and drop head is shown in dotted lines. A properly curved cap is shown in full lines, superposed thereon. The curve of this capital obviously comes outside of the reentrant angle of the depressed head, shown by the dotted line, thereby decreasing the amount of the applied bending moment by decreasing the virtual span to an extent not affected by the depressed head as will be obvious by referring to the coefficients shown in the curve of Fig. 122. The capital shown provides a gradually increasing section as the column is approached without sudden jogs or increase in the sec-



No. 127

tion. It further tends to obviate the severe shrinkage stresses which are induced in curing where a sharp junction is made between the vertical sides and the drop of a slab and wherever the capital is unscientifically designed with too little slope laterally at its junction with the slab.

Two thirds of the concrete in the depressed head is saved by a neatly curved cap and a tougher, more efficient slab results. Further, under unbalanced load a scientifically designed capital is greatly to be perferred to the clumsy drop head. Unfortunately many of the manufacturers of column forms have adopted patterns without considering or understanding the elements of efficiency. Their attention has been devoted almost solely to the question of designing something that is cheap rather than that which will impart strength to the structure.

16. Slabs Supported on Four Sides by Walls. In Chapter III we have shown the total applied bending moment is $WL/24$, where L is the span and W the total uniform load. In Chapter IX, page 235, Figs. 91 and 92, show the contours of a slab of this kind and a facsimile of the contours of the central portion of a column supported flat slab. The same principles of synclastic curvature are accordingly to be applied and for uniform reinforcement in both directions we may treat the reinforcement as 2.5 times as efficient as though it operated by mere radial resistance thus arriving at the same results that were determined by proportion in Chapter II. Relatively close spacing and small size of the bars is just as requisite in this type of slab as in the column supported flat slab and the principles deduced in Chapter VIII likewise apply.

Figuring the resisting moment of the steel according to the foregoing principles and the applied bending moment as above stated, the same section of steel is required as that under the empirical formula given in Chapter II for uniform spacing of reinforcement. This method is substantiated by computed deflections and by extensometer measurements of stresses as well.

Rectangular panels are to be reduced to the equivalent square panel by taking the side of the equivalent square as the quotient derived by dividing the sum of the squares of the sides of the rectangle by the sum of the sides.

Where the slabs are freely supported at the edges instead of being built into the walls and made rigid thereby under twisting moments the working stresses should be reduced one fifth while care should be taken not to over-reinforce the slabs as will be treated in a separate section.

Where the slabs are supported by and integral with beams, whose ratio of depth to span length is 1/10th to 1/12th, the bending of the beam imposes a somewhat greater bending moment on the slab and the twisting moments are increased with the net result that the twisting moments transfer the load nearer to the support and the intensity of the bending across the diagonal line of the panel is relieved.

Furthermore, the flatness of the curvature next to the beams is reduced by the bending of the beams at mid span along the panel edges giving a greater radius of operation for the centroid of the circumferential stresses. In a panel of this character by decreasing the spacing of the reinforcement in the middle third the efficiency of the reinforcement is increased about twenty percent over that in the wall supported panels, and may be so figured conservatively.

The distribution of the reinforcement must be uniform, that is equal symmetrical spacing normal to the two sides of the panel. If this requirement is disregarded and reinforcement put in more closely spaced in the shorter direction of the slab there is a tendency to shift the valley lines and to change the curvature of the slab so that it approaches more nearly to cylindrical curvature in the central section and the advantages of synclastic bending are impaired.

The coaction of the metal and concrete follows the same law of distribution of indirect stress as developed in Chapter VIII so that where the bars are too large in size and too far apart, the advantages of resisting in synclastic bending are greatly reduced. Furthermore, where the amount of steel is excessive in a thin slab so that the distance from the steel to the neutral plane is much less than the spacing of the bars the coaction of the metal circumferentially is largely reduced. Accordingly, if the foregoing method of computation is to be depended upon the conditions under which it is applicable must be clearly kept in mind.

We have noted that as the loads tend to develop the total resistance of the plate, cracking occurs across the diagonal lines accompanied by the rapid rise in the stresses. Reinforcement across the quarter points of the four supporting beams distributed in the bottom of the slab tends greatly to raise the yield point of the slab under heavy loading, an improvement which the arrangement described in U. S. Patent No. 1,025,236 was intended to cover.

17. Size and Spacing of Rods in Flat Slab Construction. The plate action which we have been treating, brought about by indirect stress, depends for its efficiency upon the dissemination of the steel through the mass of the concrete. Large rods and wide spacing should accordingly be avoided. Where the steel percentage is very small it sometimes happens that 5/16 or 3/8 inch rods, the smallest practical sizes, will be spaced as far apart as eight inches between centers. While reasonable results will be secured with such spacing where the percentage of steel in the belt is as small as .22, where the percent of steel is larger the spacing should be closer.

One of the common errors in flat slab design is the use of such rods as 1/2 inch or even 5/8 inch from eight to twelve inches between centers with the expectation of securing results in keeping with a more scientific and uniform dissemination of metal.

Half inch rods should not be used in slabs thinner than twelve inches at the columns, nor 5/8 inch rods in slabs thinner than fifteen inches if the best results are to be secured.

18. Flat Plate Theory. There are several important particulars in which flat plate theory diverges so widely from the ordinary considerations of beam theory that while they are in no sense contradictory of the latter when correctly understood it is nevertheless desirable that these particulars be rehearsed at such length and in such detail as to point out the nature and amount of the divergence in each case.

First, a flat plate or slab must have simultaneous curvatures of vertical sections that do not differ widely from each other, at right angles to each other parallel to the sides, both at panel center and at the supports in order for plate theory to be applicable to it. A uniformly loaded cantilever plate supported at its center only with free edges and no restraint to distribute the applied moments about rectangular axes will not usually operate in this manner but will operate as a simple beam supported at its mid point with one curvature very different from that at right angles. But when it forms part of a continuous slab of many panels it will operate in an entirely different and far more efficient manner by reason of the coaction of the two bending deformations mutually at right angles to each other, and yet the amount of the restraining force that must be called into play to prevent plate action from degenerating into mere beam action is nominal only.

The fact of the wide difference between the action of a floor as a plate and as a beam is implicitly recognized in the common limitation requiring the panel width to be at least 75 percent of the panel length. Smaller widths tend to give rise to beam action rather than plate action.

Suitable provisions must in all cases be made to ensure plate action and prevent it from degenerating into beam action.

In slabs it is necessary to provide a wide and fine-grained reinforcement to fully cover the areas of synclastic curvature, such as will make them each offer a continuous uniform resistance to bending in all directions.

It is to be noticed that the ratio of slab thickness to span is one of the important factors in preserving of plate action. Were floor slabs in reality thin plates with a thickness of less than one percent of the span the problem of securing permanent plate action under unbalanced loads would no doubt be the most crucial one to be met in the practical realization of plate action, whereas, when the slab is restrained by stiff columns, plate action can be ensured in interior panels with relatively little special provision compared to that required in exterior panels.

Second, the total panel area has been subdivided by sections of zero vertical shear into load areas such that it is known to which of the supports each such load is carried, and furthermore, the intensity of the vertical shear on each plane within that area by which the load is carried to its support is also known. Such sections of zero shear lie along the sides of the panel, along the diagonals, and along the median lines. No vertical forces are transmitted across these sections and the loads in the triangular areas so bounded are transmitted by vertical shears to the columns at their apices. These shears are in each case resisted solely by the material within the triangular area, as they must be because each area is completely bounded except at the column by sections of zero shear.

These known shears completely determine the total applied bending moment in the panel, as may be seen from the following considerations: The total shear S_x, See Fig. 123, acting on a vertical section *PF* perpendicularly to x, in transmitting a vertical force equal to S_x thru a distance dx forward along x, whether it be at the same time transmitting it thru distance laterally and so perpendicularly to x or not, must necessarily generate an element of bending moment $dM_x = S_x dx$ along x. This result necessarily follows from the principles of statics and the definition of bending moments. And the elements of bending moment so generated may be regarded as by-products and necessary accompaniments of the primary phenomenon of the transmission of the load to the supports by vertical shears, the transmission of the shears being the primary physical cause of the bending.

It has become so habitual to reckon bending moment from considering the statical moment of the load and its arm, that the fundamental operation of transmitting load by shear has been to some extent disregarded. But in the present case it is evident that no part of the load that is not transmitted by shears perpendicular to x produces moments along x, and the part so transmitted is fixed by the planes of zero shear that bound it.

When each of the eight triangular areas that surround a support have been considered, it is evident the entire load W upon all of them together is transmitted to that support alone, one half of it by vertical shears across sections perpendicular to x that extend from diagonal to diagonal, and one half of it across similar sections perpendicular to y. No total bending moments along x and along y can be produced except those generated by these shears because these shears transmit the entire load to the support.

It has been shown that each of the total moments at mid span and margin amounts to $WL/12$ about a side when the support is a point, or to $WL\sqrt{2}/16$ about a diagonal, and is much less in the practical case of a cap of the usual size. It appears therefore that the total applied moment at mid span and support is less than $WL/16$.

The peculiarities of the separate transmission of triangular loads of rectangular panels to the supports are due to the arrangement of the supports in rectangular array along x and y. An arrangement of supports located at the corners of equilateral triangles for example would require an entirely different set of sections of zero shear, which could be readily discovered from considerations of symmetry, and when once found they would necessarily determine the magnitude of the total applied bending moments along the several axes in the slab.

Altho the vertical shears are confined to the areas where they are generated, the longitudinal stresses arising from the bending moments produced by their shears are not so confined but are distributed along the sections at mid span and supports on the principle of rigidities.

Of special signifiance are the sections of zero shear in a floor which is reinforced with narrow belts of rods that run directly and diagonally from column to column, since each belt then acts largely independently of the others in transmitting its load to the support much as would a separate beam.

Third. In floors which are continuously reinforced with wide spreading and fine grained mats embedded in the concrete in such a manner that they afford the necessary resistance to the tensions thruout their synclastic areas, and thus make them imitate in their mode of operation the action of continuous plates, the steel in the belt that crosses synclastic areas is not subjected to tensile stresses by the applied bending moments which can be correctly calculated in the same manner as they are in the longitudinal reinforcement of beams. The tensile stresses in the steel are in reality less than one half as great as those so calculated, a fact substantiated by test measurements and susceptible of mathematical demonstration as was shown in Section 9 Chapter VIII somewhat in detail but may be expressed in abbreviated form by saying that in attempting to bend a plate or slab that is in a state of synclastic or convex curvature it will act like a piece of a shell, and any bending about one axis is physically so related to that about a second axis at right angles to the first axis that any increase of curvature about the first axis is necessarily accompanied by a corresponding decrease of curvature about the second axis. And that a given total applied bending moment can produce no more than a certain amount of bending in the shell about both axes taken together which when equally subdivided between the two is only half as great in each as if the entire bending were about a single axis as in ordinary beam theory.

But bending is deformation, and to cut total deformation in two means that stresses as expressed by extensometer measurements of elongation are also cut in two. The fact of the physical connection between the increase of curvature about one axis and the decrease about an axis at right angles to it is intimately connected with the relative incompressibility of the material on the compressed side of the shell, and its resistance to specific increase of area of the mat on the tension side of the shell. Any elastic compression along x finds relief by a corresponding elongation along y, to a larger extent than by a change of volume.

Fourth. A column cap of a given size, does not exert an influence of the same amount in reducing the total applied bending moment in flat plate theory as it does in beam theory, a fact which is evident from the shape of the moment curve which is different in a beam from that on either a side or diagonal of a panel. The large cap required in slab floors make the consideration of their size one o the essential points in any correct calculation of the applied mo ments and this is a matter that has been fully and correctly treated for the first time in this treatise and the results stated in convenient tables.

Fifth. Owing to the diversity in the arrangement and amount of the steel in the synclastic areas over column heads and around panel centers the moments of inertia at mid span and supports usually differ materially from each other. But the relative magnitude of these moments of inertia is the determining factor in fixing the position of the points of inflection on the sides and diagonals of the panel, and in subdividing the total applied bending moment between mid section and supports, which ordinarily differs greatly from the usual assumption of beam theory with constant moment of inertia. This question also has here been fully investigated and expressed in tables for the first time.

CHAPTER XI

REINFORCED CONCRETE COLUMNS

1. General Considerations. The requirements for suitable design for reinforced concrete columns in building construction may be briefly stated as follows:

First, that the longitudinal reinforcing metal should be toward the outer portion of the column in order to properly resist any tendency to bend or deflect.

Second, that the bars should be banded or tied together to maintain them in their desired position and add toughness to the column.

Third, that the bands or ties should not so cross the core as to interfere with placing the concrete and securing a monolithic solid core.

Fig. Type A, shows one of the old forms of Hennebique type of column. In this type the principal reinforcement consists of heavy vertical or longitudinal bars tied across laterally from one to the other by comparatively small ties. It will be noted that these ties cross and recross the core of the column, and require considerable care in filling to make sure that there are no voids in the finished work.

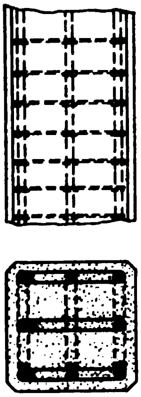


Fig. Type A Column

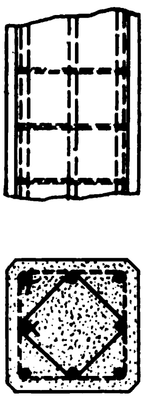


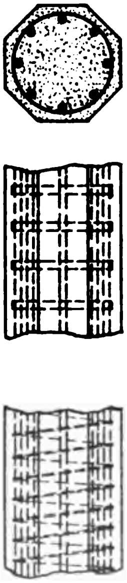
Fig. Type B Column

Cases have occurred with this type where the concrete has been arrested part way down in pouring the column and on removal of the form an open space was found of perhaps as much as two feet between the concrete above and below, so that the load above was carried by the vertical bars only. Evidently such an arrangement of metal is somewhat dangerous, but with unusual care it may prove satisfactory from the standpoint of strength if the work be properly executed.

Fig. Type B, shows an improved form of tying together the eight vertical bars forming the vertical reinforcement with horizontal ties in the form of squares, one inscribed within the other. The advantage of this type over that previously shown lies in the fact that the central core of the column, or inscribed square, is clear and unobstructed thruout.

Fig. Type C, shows a column reinforcement consisting of four vertical rods with wrapping or ties holding them together at intervals. This is suitable for very light loads where the concrete is more than sufficient to take the entire compression without excessive stress.

Fig. Type D, shows a column section of the Considère type in which the vertical rods are hooped with spiral reinforcement. Considerable work has been executed using hooped columns that omit the vertical steel. This, as the authors view it, is a very grave mistake.



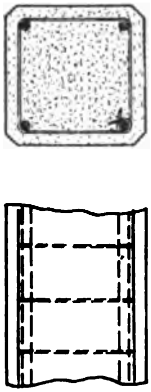


Fig. Type D Column Considère Type. Fig. Type C Column Suitable for Light Loads Only.

Hooping may be of two types: First, a spiral coil in which the wire is wound around the core of the column in the form of a continuous spiral, and second that in which separate independent hoops are placed at intervals and attached to the vertical reinforcement.

The strength of Types A, B, C, and D, all depend, first upon the strength of the concrete, second, upon the amount of vertical steel used, and, third, upon the amount of ties or hooping holding the rods in position and bringing lateral restraint upon the concrete.

Theoretical formula based on the ratio of the moduli of elasticity of concrete and steel alone cannot be depended upon for a satisfactory solution of the problem presented by the third element noted and we must depend largely upon experimental investigation to determine reasonable and safe practical values to use for our working stress.

In deciding upon these values we need to consider the column, first, from the standpoint of its ultimate strength in the finished building, second, from the standpoint of its strength and safety during construction, and third, from a consideration of the relative values of the various types in securing strength at a minimum cost.

Type D, with a proper proportion of vertical steel combined with the hooping ranks first, from the standpoint of safety and economy.

Type B, second.

Type D, with hooping but with no vertical steel third; and types A and C fourth.

It may be stated that type A is now rarely used and discussion concerning it may be omitted.

For Type C, the allowance permissible for working stress on a 1 : 2 : 4 concrete is 350 pounds per inch of the core area between rods, and 10,000 pounds per square inch on the vertical steel and besides this the volume of metal in the ties is to be treated as forming imaginary verticals with a working stress equal to that allowed for the vertical bars, the ties to be spaced not further apart than ten times the diameter of the vertical bars in case the bars are one inch section, but where smaller bars are used the spacing should not exceed 9″ nor the size of the tie to be less than one quarter inch round.

Type B. The allowable working stress for a 1 : 2 : 4 mix is 600 pounds per square inch on the concrete of the core, 10,000 pounds on the vertical steel, and one and one half times the volume of the ties treated as imaginary verticals. These ties should not be spread further apart than 9″, and, if they are to be considered of value, they should be put not more than ten diameters of the vertical bars apart.

Type D, the Considère type, is by far the most economical type of column reinforcement that has been invented. It was brought prominently to the attention of the public by Armand Considère.

The principle involved is this: by restraining the concrete laterally its strength in compression is greatly increased. Just as an ordinary piece of stove pipe filled with sand will carry a load several times greater than the pipe itself would be able to do, so will a hooped column owing to the fact that the metal is strained in tension, while the filling held in position by the restraint of the pipe, carries the weight of the load. For strength see Section 5.

There have been quite a number of experiments on hooped concrete using spiral hooping only. In these experiments it has been found that after the ultimate strength of plain concrete has been developed, splitting and scaling of the outside shell occurs, combined with a large vertical deformation and considerable lateral bending before ultimate failure.

2. Considerations of Safety Determining Carrying Loads. If it is expected to develop the core of the concrete to a point beyond its normal strength we must evidently prevent its lateral distortion or bulging and also the sliding or flow of the concrete between consecutive bands or turns of the spiral, hence a certain proportion of vertical steel must be used in connection with the hooping to secure the best results.

In determining the degree of safety of the various types of column design, an investigation of the manner of failure of the respective types is in order as to whether it occurs suddenly and without warning, or gradually, accompanied by indications of approaching failure long before failure occurs, and there is the further question as to whether the conditions of strain in the column are proportional or comparable between the column under ordinary working stresses and the column as it approaches the breaking down point and ultimate strength. The following general observations may be made as to these questions:

In columns shown in Fig. Type C, the failure occurs with little warning, the vertical bars bending outward and the ties yielding.

In the hooped column without vertical steel, when it is loaded from forty to fifty percent of its ultimate strength, the portion of the concrete outside the hooping commences to check and crack, and later to scale. From this point, the rate of deformation with addition of the load increases rapidly owing to dissipation of energy by the cracking and scaling of the shell. Further loading is accompanied by large lateral deformations up to the final failure. Such a column gives ample warning but the point at which the outer shell or fire protection commences to fail is but little higher than the point at which the ordinary vertically reinforced column fails, so that little advantage in the way of increased working stress is secured unless the hooping is combined with vertical steel.

The well hooped column vertically reinforced shows a large increase in strength over that of the vertically reinforced column with ties and a great increase in toughness. Its failure is not sudden and without warning as in the former type, while the point at which checking and scaling of the outside shell occurs is raised to eighty or eighty-five percent of the ultimate strength, thus giving a large margin of safety to the fireproofing between the working load and the load where the failure of the shell is in evidence.

3. Experimental Data. A partial report on tests on full sized columns, made at Phoenixville, Pa., for C. A. P. Turner, engineer, by Mason D. Pratt, is given in the following table:

Test No. 1

Marks on column: None.

Reinforcement: Eight $1\frac{1}{8}$ inch round vertical bars.

Bands: Spaced 9 inches vertically, $\frac{1}{2}$ inch rivets, cross section $1\frac{3}{4}$ x $\frac{1}{4}$ inches, inside diameter 14 inches.

Hooped with 7/32 inch wire spirals, about 2 inch pitch.

Total load at failure, 1,360,000 lbs.

Remarks: Point of failure was about 22 inches from the top, Little indication of failure until ultimate load was reached.

Some slight breaking off of concrete near the top cap, due possibly to the cap not being well seated in the column itself.

Test No. 2

Marks on columns: Box 4.

Reinforcement: Eight $1\frac{1}{8}$ inch round vertical bars.

Bands: Spaced 13 inches vertically, $\frac{1}{2}$ inch rivets, cross section $1\frac{3}{4}$ x $\frac{1}{4}$ inches, inside diameter 14 inches.

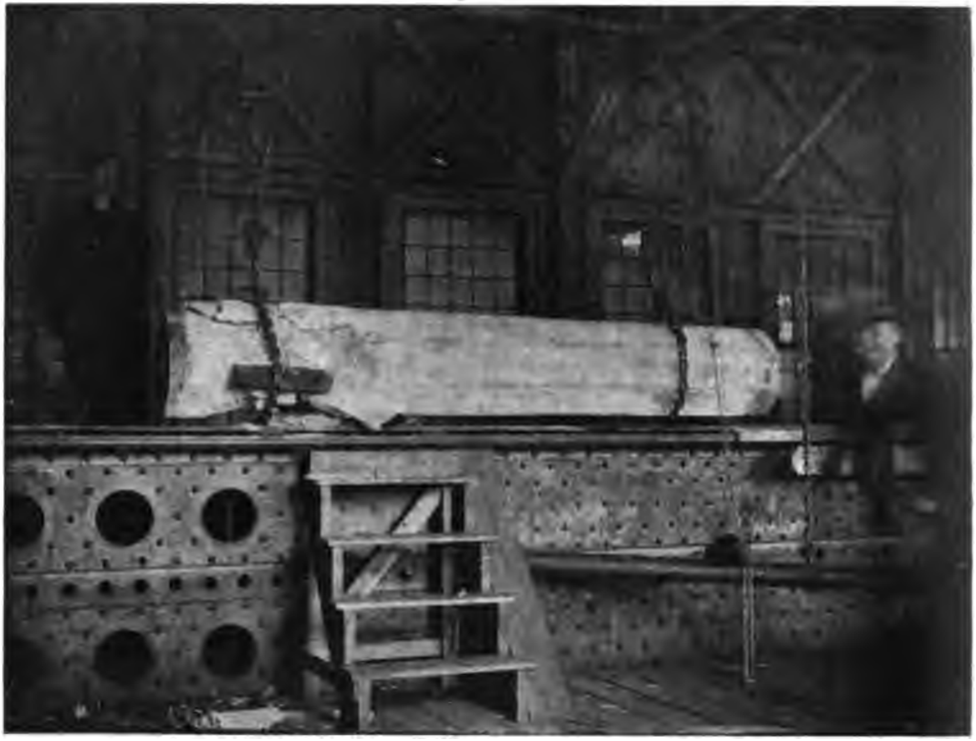
Hooped with 7/32 inch wire spiral, about 3 inch pitch.

Point of failure: About 18 inches from top.

Top of cast iron cap cracked at four corners.

Ultimate load: 1,260,000 lbs.

Remarks: Both caps apparently well seated, as was the case with all the subsequent tests.



Test No. 1
Showing Column as it Came from Testing Machine.

Test No. 3

Marks on column: 4-b.

Reinforcement: Eight 7/8 inch round vertical bars.

Bands: Spaced 13 inches vertically, ½ inch rivets, cross section 1¾ x 3/16 inches, outside diameter 14 inches.

Ultimate load: 900,000 lbs.

Point of failure: About 2 feet from top.

Remarks: Concrete at failure, considerably disintegrated, probably due to continuance of movement of machine after failure.

Test No. 4

Marks on columns: Box 4-c.

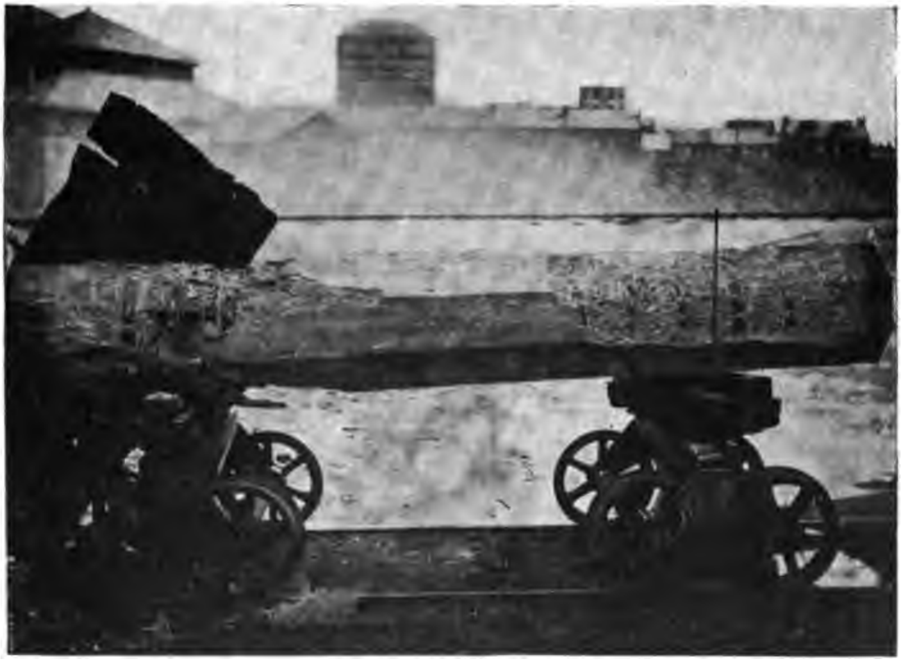
Reinforcement: Eight 1 inch round vertical bars.

Bands: Spaced 8 inches vertically, ½ inch rivets, cross section 1¾ x ¼ inches, inside diameter 14 inches.

Hooped with 7/32 inch wire spirals, about 3 inch pitch.

Total load at failure: 1,260,000 lbs.

Remarks: First indications of failure were nearest the bottom end of the column, but the total failure was as in all columns, within 2 feet of the top. Large cracks in the shell of the column extended from both ends to very near the middle. This was the most satisfactory showing of all the columns, as the failure extended over nearly the full length of the column.



Column No. 4, after Test

Test No. 5

Marks on column: None.

Reinforcement: Eight ⅞ inch vertical bars.

Bands: Spaced 10 inches vertically, ½ inch rivets, cross section 1¾ x ¼ inches, 14½ inches outside diameter.

Hooped with 7/32 inch wire spiral as before, 3 inch pitch.

Load at failure: 1,100,000 lbs.

Ultimate load: 1,130,000 lbs.

Remarks: The main point of failure in this as in all other columns was within two feet of the top altho this column showed some scaling off at the lower end.

This set of tests were not conducted with any considerable degree of refinement but were a practical test of ultimate strength and the yield point value of specimens of full sized members, which lends greater value to the determination than laboratory tests on small specimens.

The concrete mixture was one part Portland cement, one part sand, and one and one-half parts buckwheat gravel, and three and one-half parts gravel ranging from one-quarter inch to three-quarter inch in size.

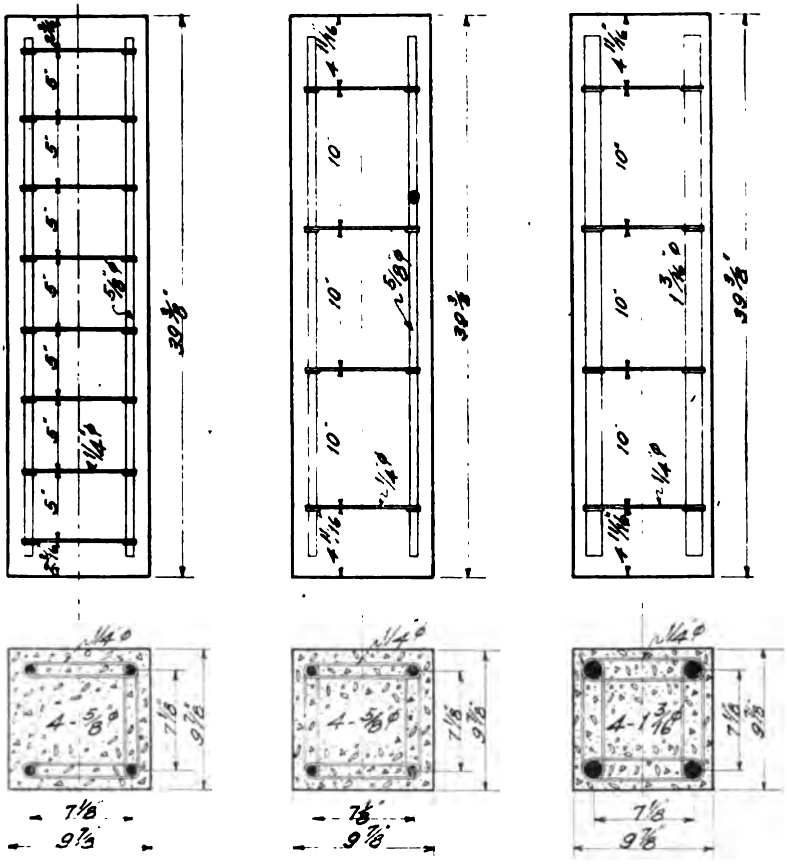
It should be noted that in these tests the cracking of the shell did not occur until the hoops were over-strained, and that the strength of the hooping closely defined the ultimate strength of the column with the proportions of vertical steel used.

1. In these columns, pressures were developed on the core more than three times the ultimate strength of plain concrete at 2600 pounds per square inch.

2. Incipient failure occurred only by the stretching or bursting of the bands.

All columns were approximately octagonal in shape, 10′6″ long and 18″ diameter. Final failure occurred toward the upper end of the column. Mr. Pratt accounts for the regularity with which the columns failed at the upper end on the ground that the concrete at the lower end was more dense, owing to its being under considerable hydraulic pressure, while setting. The rods were all shorter than the concrete shaft. Examination of the columns after removal from the testing machine showed in all cases a bulging out of the vertical reinforcement at the principal point of failure with the nearest hoop ruptured and in every case the wire spiral was broken in one or more coils at the point where the vertical rods were bent out. The vertical bars in nearly every case bulged as a column with fixed ends. Where the hoop spacing was six to nine inches, the deformed length of the bar would extend over the space of two hoops. Where the hoop spacing exceeded nine inches the deformed length of vertical bars was confined to the space between one pair of hoops.

It may be well to review at this point two series of carefully conducted tests upon plain and reinforced concrete prisms. Fig. 127, shows the dimensions and reinforcement of a series of prisms tested by Bach* for Wayss and Freitag. These specimens are Type C columns.



Test Specimens

Fig. 127 Showing Dimensions and Reinforcement.

The first table for elasticity test of Type C columns, shows the reinforcement, tie spacing, and elastic deportment under different loads.

The second table for Type C columns, shows the breaking strength of this series.

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*See Het Cement-Ijzer, Saunders. p. 91.
Eisenbetonbau, Mörsch.

It will be noted under the table showing the breaking strength that the specimens with 1 3/16 inch round verticals and 4.6 percent reinforcement, do not show as great an increase in strength as the specimens with 1.14 percent reinforcement, but with much closer spacing of the ties, the specimens with the large rods having ties four times as far apart as the specimens with the small rods.

COL. TABLE I

Elasticity Test of Columns (Bach)

Stresses	4 Rods of Diameter	Tie Spacing	Shortening in Millionths of the length		
Lbs. sq. in.	Inch	Inches	Total	Elastic Diff.	Permanent Set
459	Plain Concrete		133	7	126
459	⅝ in. rd.	10	114	5	109
459	⅝ " "	5	110	2	108
459	⅝ " "	2½	106	4	102
919	Plain Concrete		333	37	296
919	⅝ in. rd.	10	267	20	247
919	⅝ " "	5	264	18	246
919	⅝ " "	2½	241	13	228
1380	Plain Concrete		709	164	545
1380	⅝ in. rd.	10	488	63	425
1380	⅝ " "	5	473	58	415
1380	⅝ " "	2½	421	42	379

COL. TABLE II.

Breaking Strength of Columns (Bach)

Specimen 3 mo. old.		Breaking Strength				
Diameter of rods.	Tie Spacing	Each			Average lbs. per sq. in.	Percent of Reinforcing
Plain Concrete		2076	1963	1977	2006	0
⅝ in. rd.	10 in.	2432	2290	2447	2390	1.14
⅝ " "	5 "	2389	2660	2489	2512	1.14
⅝ " "	2½ "	3015	2845	2887	2915	1.14
¾ " "	10 "	2404	2404	2446	2418	2.04
1 3/16 " "	10 "	2474	2830	2802	2702	4.60
Test Cubes.		2389	2404 2631	2432 2617	2490	0

Looking at the elasticity tests of the specimen reinforced with the same vertical steel the marked difference in toughness, increase of modulus of elasticity and the reduction of permanent set produced by the increase in the number of ties, is strikingly shown by this series of tests.

This series of tests show conclusively that where the vertical bars do not bear directly upon the face plate of the machine and the load is brought upon the reinforcement under the usual condition of the column in the reinforced concrete structure, that is the steel is strained by the load brought on it thru the concrete, that a formula which does not take into consideration the amount and spacing of the ties fails to account for the deportment of the column both as regards elasticity and ultimate resistance. That is, merely an increase in the total cross section of the longitudinal reinforcement does not produce an increase in the breaking strength to the extent which would be expected by the formula

$$P = f_c\,(A_c + nA_s)$$

Hence in inexperienced hands, this formula may produce unsafe designs by increasing the percentage of longitudinal reinforcement disproportionately in order to secure columns of small diameter. This procedure gives a column with a calculated margin of safety which it does not possess.

When the increase in the resistance is compared for equal longitudinal reinforcement, there being a difference in the number of ties, it is shown by these experiments that the steel used as ties is much more effective than the longitudinal steel. As noted, however, the columns reinforced with vertical steel and ties do not develop that degree of strength which is secured by columns properly reinforced vertically and hooped with bands or spirals. Hence their use has been largely discontinued where loads are at all heavy and is confined to those cases where loads are relatively light and some slight saving may be made in the use of ties over the cost of spiral hooping.

4. Reinforced Columns Classified by the Manner in Which Loads are Applied. We have called attention to the type of prisms which were tested by Bach in which the steel was cut short at the end of the concrete prism so that the load applied to the prism was brought upon the longitudinal steel thru the bond shear between the steel and the concrete. This is the common mode of arrangement of the steel in the reinforced concrete column in the practical building. Sometimes, however, in order to secure relatively small columns for heavy loads very high percentages of vertical steel are used and the load is brought largely upon the steel by direct bearing of steel upon steel. Evidently the coaction of the elements of the composite structure differ widely under these radically different conditions and correspondingly different formulas must be applied to these different conditions.

We have cited the interesting series of tests by Bach on rectangular prisms, plain and reinforced with longitudinal steel and ties and we may profitably consider the corresponding series of tests on hooped and longitudinally reinforced prisms.

Fig. 128 gives the dimensions and type of reinforcement used while table III shows the results of the tests.

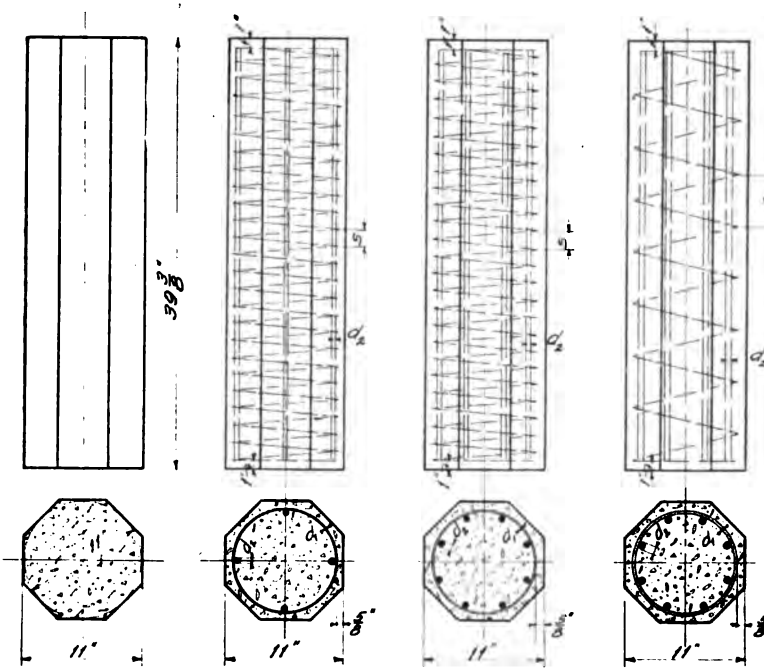

Fig. 128 Dimensions of Columns and Type of Reinforcement.

As in the first series the longitudinal steel was cut short of the ends of the prisms so as to secure normal coaction between the steel and the concrete. Percentage of hooping varied thru a wide range and the pitch of the spiral hooping was also varied between wide limits. Unfortunately the series is lacking in a corresponding variation in the percentage of vertical steel.

On the whole, these specimens correspond approximately with results indicated by the Considère formula and exceed these results for those specimens which approach desirable proportions in point of spacing of the spiral and relation of the vertical steel to the hooping.

COL. TABLE III

Bach's Tests of Plain and Hooped Concrete Prisms, Longitudinal Steel Cut Short of End of Prism.

Col. No.	Description	Spirals		Verticals		Area Core sq. in.	% Reinforcement			Load lbs. per sq. in. when first cracks started.	Increased st. due to reinf. Lbs. per sq. in. at yield pt.	Increased st. lbs./sq. in. per 1% reinf. at yield pt.	Ultimate strength lbs. per sq. inch	Increased ult. st. lbs./sq. in. due to reinf.	Increased ult. st. in. lbs. per sq. in. per 1% reinf.	Load when considering core only lbs./sq. in.
		Dia. wire in.	Pitch	No.	Dia. in.		Spiral	Vert.	Total							
I	4 spec.	Unreinforced					0	0	0	1,890			1,890			
II	3 "	.2	1.52	4	.28	69.69	.88	.35	1.23	2,260	370	301	2,260	370	301	3,270
III	3 "	.28	1.48	4	.28	68.51	1.78	.36	2.14	2,290	400	187	2,550	640	299	3,660
IV	3 "	.4	1.68	4	.28	66.76	3.24	.37	3.61	2,420	530	147	3,420	1,525	422	4,930
V	3 "	.2	1.52	8	.44	69.69	.88	1.75	2.63	3,190	1,300	495	3,210	1,320	502	4,650
VI	3 "	.28	1.48	8	.44	68.51	1.78	1.77	3.55	3,270	1,380	389	3,270	1,380	389	4,720
VII	3 "	.4	1.72	8	.44	66.76	3.17	1.82	4.99	3,460	1,570	315	4,000	2,110	424	5,770
VIII	3 "	.28	1.24	4	.28	68.51	2.12	.36	2.48	2,790	900	363	2,850	960	387	4,100
IX	3 "	.40	1.60	4	.28	66.76	3.41	.37	3.78	2,420	530	140	3,000	1,110	294	4,340
X	3 "	.48	1.64	4	.28	65.61	4.82	.38	5.20	2,560	670	129	3,640	1,750	336	5,260
XI	3 "	.56	1.48	4	.28	64.47	7.36	.38	7.74	2,250	360	47	3,500	1,610	208	5,050
XII-I	3 "	.28	1.60	8	.20	68.51	1.64	.37	2.01	2,320	430	214	2,320	430	214	3,360
XII-II	3 "	.40	1.60	8	.28	66.76	3.41	.74	4.15	2,330	440	106	3,270	1,380	333	4,720
XII-III	3 "	.56	1.60	8	.40	64.47	6.82	1.56	8.38	2,620	730	87	4,290	2,400	286	6,200
XIII-I	3 "	.28	3.20	8	.28	68.51	.83	.72	1.55	2,300	410	264	2,300	410	265	3,330
XIII-II	3 "	.40	3.20	8	.40	66.76	1.71	1.51	3.22	2,550	660	205	2,580	690	214	3,720
XIII-III	3 "	.56	3.20	8	.48	64.47	3.43	2.25	5.68	2,650	760	134	2,830	940	166	4,240
XIV-I	3 "	.28	4.80	8	.40	68.51	.55	1.47	2.02	2,200	310	153	2,210	320	158	3,190
XIV-II	3 "	.40	4.80	8	.48	66.76	1.15	2.17	3.32	2,600	710	214	2,600	710	214	3,750
XIV-III	3 "	.56	4.80	8	.56	64.47	2.30	3.11	5.41	2,950	1,040	193	2,940	1,050	195	4,250

Specimens had an octagonal section with a diameter of 10.8 inches and had a length of 39.37 inches. Mix—1 part of Heidelberg Portland Cement to 4 parts of Rhine Sand and gravel. Specimens 5 to 6 mo. old when broken.

Specimens XIII and XIV, in which the pitch of the spiral was exaggerated gave results less than the formula, pointing to the necessity for closer spacing.

In specimens II, III, IV, VIII, IX, X, XI and XII, the sectional area of the longitudinal rods was small and the results were consequently indifferent, but the greater the total weight of spiral reinforcement the higher were the values.

The tests show that when the spirals are increased in strength, their pitch must be decreased and the cross section and number of vertical rods must be increased, for with the increase of spirals the concrete is in a condition to resist heavier pressure and its tendency to force its way out between the longitudinal rods and the spiral bands increase.

Examining the table of the values obtained, the increase in strength and yield point value per one percent of total steel is found to be greatest in column specimen V, in which the spiral reinforcement is .88 percent and the vertical reinforcement was 1.75. In columns XII and XIII, with the spiral reinforcement 6.82 percent and the vertical reinforcement 1.56 percent, the increase percent of total reinforcement at the yield point was only 87 pounds and the increase of the ultimate strength per percent of the ultimate reinforcement 286 pounds per square inch, showing an improper proportion of the spiral and vertical reinforcement.

On noting the pitch of the spiral it is quite noticeable that the specimen with a small pitch gives the highest resistance per percent of steel. In none of these specimens, however, are the longitudinal rods of sufficient size to secure the best results. Had 5/8, 3/4 and 1 inch rods been used in place of 1/4 inch, 7/16 and 1/2 inch, much higher values would have resulted from the stiffness of the verticals which must act as beams from coil to coil in resisting outward or bulging pressures.

The following table gives the percentage of vertical steel and hooping in the Phoenixville tests, see Section 3.

COL. TABLE IV

Phoenixville Column Tests. Specimens 3 mo. old

No. of Test	% Verticals	% Spiral	% Hooping	% Spiral and Hooping	% Total	Ultimate Tested Strength	Ultimate Strength Per Sq. In.	Computed Increase per % of Steel	Strength by Considère Formula
1	5.17	1.54	.99	2.53	7.70	1,360,000	8,850	785	1,291,000
2	4.82	1.35	.66	2.01	6.83	1,260,000	7,630	710	1,279,000
3	3.12	0	.72	.72	3.84	900,000	5,800	792	898,300
4	4.08	1.36	1.12	2.48	6.56	1,260,000	8,200	820	1,216,000
5	3.12	1.36	.90	2.26	5.38	1,130,000	7,350	842	1,127,000

From these tests it may be observed that the best results are to be secured when the value of the vertical steel as an element of strength is approximately the same as the hooping according to Considère's formula, Section 5. In other words, the reinforcement should be so proportioned that the volume of the hooping for this type of column should be between the limits of 35 and 45 percent of the volume of the vertical steel to secure the best results in increasing the yield point and ultimate strength of the column. Between these limits the yield point value of the columns should vary from 75 to 85 percent of the ultimate strength of the specimen, so that due warning of approaching failure is given by the member while ample margin exists between the safe working stress and that point at which the fireproofing will commence to scale and chip.

5. Considère Formula. As we have pointed out, the Considère formula is conservatively applicable to the vertically reinforced and hooped columns provided that the hooping and vertical steel are properly proportioned. With no vertical steel, the yield point value of the column is not sufficiently raised to warrant very material increase in safe working stress over that of ordinary types notwithstanding the fact that there is an increase in the ultimate resistance and large deformations occur before ultimate failure.

The Considère formula for column resistance is as follows:

$$P = 1.5\, A_c f_c + (f_s A_s + 2.4 f'_s A'_s)$$

in which A_c is the area of the core, A_s the area of the longitudinal steel, A'_s is the area obtained by dividing the volume of the hooping by the length of the column. The coefficient 1.5 is a coefficient which Considère considers represents the effect of the hooping in increasing the strength of the core, and that this coefficient is a maximum at 1.5 and that it is less than this value for a percentage of hooping which does not furnish a resistance equivalent to 700 pounds per square inch lateral pressure on the column.

Mörsch, however, and also Saunders, in discussing the Bach tests where the percentage of steel was low, seem to treat this coefficient of Considère as the ratio of the core to the total area of the column (fireproofing and all). In most building ordinances this coefficient, however, is taken as unity. 2.4 times the volume of the hooping is figured as the effect of the stress on the hooping in increasing the crushing resistance of the core by the lateral pressure brought about by the stress in the hooping and represents the resistance which would be developed were the hooping in the form of cylinders and filled with a granular mass such as sand subjected to pressure.

Hence $2.4A'_s$ equals the cross section of the equivalent imaginary verticals.

Le Genie Civil, Feb. 9, and 16, 1907, reported a very extensive series of tests by Considère on 260 columns with lengths varying from 3.15 in. to 13 ft. 1½ in. and diameters from 1.8 in. to 27.5 in. The percentage of reinforcement varied from 1 to 14, various methods of spiral reinforcement, including concentric spirals were tried. The effects of richness of mixture, age, percentage of water, ramming and irregularities in workmanship were also studied. One specimen having 12.9 percent spiral and 1.2 percent longitudinal reinforcement, and made from a mixture containing 1830 lb. of cement per cu. yd. of sand sustained 25,600 lb./in.^{2} at rupture. A similar specimen having only 6.6 percent of spiral reinforcement failed at 17,200 lb./in.^{2} at rupture, with a deformation of 12 percent. From these and previous tests Considère deduced the conclusions that the rupture load of a spirally reinfored concrete column exceeds the sum of the three following factors:

1. The strength of the plain concrete times 1.5.
2. The strength of the longitudinal rods stressed to their yield point.
3. The strength of a longitudinal reinforcement of 2.4 times the weight of the spiral stressed to its yield point.

Turner, in the design and guarantee of strength for working several hundred thousand columns has used the Considère formula slightly modified,allowing 800 pounds per square inch on a 1: 2: 4 mix, 12,000 pounds per square inch on the vertical steel, 16,000 pounds per square inch on the hooping, treating the spirals as imaginary verticals having a volume 2.4 times the volume of the hooping. For these values the ratio of the length of the column to its diameter should not exceed 12. For a longer column increased vertical steel should be used to resist flexure. Where the vertical resistance of the concrete developed by the hooping exceeds 2000 pounds per square inch the proportion of the mix should be increased from 1 : 2 : 4 to 1 : 1½ : 3, and extra care should be used in the selection of the stone aggregate to see that it is hard and satisfactory. Screened gravel is preferable where high working pressures are used in the columns. To be efficient, spiral hooping should not exceed a 4″ pitch when combined with not less than four vertical rods and these should be limited to a minimum diameter of 3/4 of an inch for columns carrying moderate loads, or to a pitch not exceeding 3″ if the full value above recommended is to be used in considering the hooping. A reduction of the allowance on the vertical steel should be made where the spacing of the vertical bars exceeds 9″ center to center in proportion to fifty percent of the increase in spacing of the bars.

6. Safe Ultimate Limit of Compression. A wonderful degree of strength which may be developed by properly hooped and longitudinally reinforced concrete columns is established by the Considère experiments and it becomes a question as to how great values it is permissible to use. Turner has found it permissible, where small diameters are desired, to use as the approximate limit of safe design a working stress as high as 4000 pounds per square inch of cross section of the core including the vertical steel. Under these ultimate conditions it is recommended that the column should have enough vertical steel to carry the entire load at little more than the yield point value, say at 50,000 pounds per square inch; that there be sufficient hooping to develop the value of the load figured at

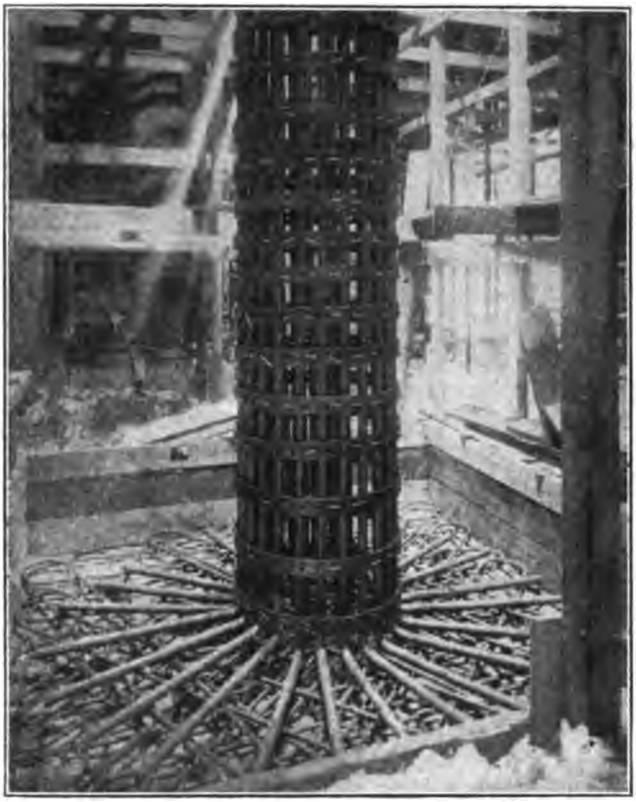

Fig. 129. Cut showing reinforcement of column carrying working pressure of 4000 pounds per inch on the core.

40,000 pounds per square inch on imaginary verticals corresponding to 2.4 times the volume of the hooping; and that the gross area of the column inside the fireproofing should be sufficient to carry the load at 4000 pounds per square inch.

7. The Mode of Operation of the Reinforcement in Concrete Columns. In a practical concrete structure, as a rule, the load is brought on the column thru the beams or the concrete of the slab and the load is transferred to the vertical steel thru the medium of the concrete by the bond between the concrete and the steel.

In the tests by Withey and those of some other American investigators, the load on the test specimens was transferred directly to the steel by direct bearing between the longitudinal reinforcement and the face plate of the testing machine. This makes an important difference in the mode of operation of the column. The vertical steel is a more rigid element under compression than the concrete and if the load be not applied directly to the steel but be brought on the steel thru the surrounding matrix, the coaction of the two materials is brought about by bond shear and indirect stress generated by the bond shear between the steel and the concrete. Now, since there is no slipping of the steel in the concrete within the yield point of the column, the same amount of potential energy or work of deformation is stored up by the indirect stress of the bond shear between the concrete and the steel as is stored directly in the steel. Hence it appears that in this case the steel by help of bond shear operates to store energy more efficiently up to the point where cracks and checks occur in the concrete than a mere comparison of the relative moduli of steel and concrete would indicate.

This phase of the co-operation of the steel and the concrete in the column requires, for its greatest efficiency, a strong, rich, concrete, in order that the most dependable bond may be secured.

If, however, the load be brought upon the steel by a rigid bearing between the face plate of the testing machine and the steel of the longitudinal reinforcement and in like manner on the concrete core, such a bearing makes the elastic deformation of the steel the same as that of the concrete thruout, and no elastic efficiency by coaction of the concrete and metal exists, such as occurs in the practical structure.

8. Effect of Hooping. The crushing of solid bodies cannot take place without lateral swelling. Therefore, by resisting any such swelling, the compressive resistance of the column is increased. In the practical column this resistance to lateral bulging or swelling is furnished by the hooping in which the spacing should be limited generally to four inches for light pressures and closer spacing for higher pressures. As the hooping is brought into play by lateral swelling or bulging, the degree of restraint furnished by the hooping alone is not uniform and this lack of uniformity causes the concrete to check and crack outside of the hooping under pressures as large approximately as the ultimate strength of plain concrete.

The addition of vertical steel distributes the bulging pressure from band to band or hoop to hoop to such an extent that the vertical steel which has been added forms beams spanning the spaces between coils or bands and does so to an extent measured by the more or less close spacing of the vertical bars. In other words, the vertical reinforcement receives lateral pressures between the hoops and transfers it to the hoops as supports. This action destroys the equilibrium of uniform pressure outward upon the hoops and tends to deform the configuration of the circular hoop from circles to polygons having apices at the point of bearing of the vertical steel against the hoops. This action between the vertical steel and the hooping induces indirect stresses between the hoops and the vertical bars similar to those in the slab with two way reinforcement.

The outer shell of the concrete is subjected to direct compression vertically and circular tensions horizontally, brought about by the bulging tendency. These circular tensions are reduced between verticals by the compression brought about by the tendency to change configuration in the hooping just mentioned, while the ring tension opposite verticals is increased by this action. The lateral reinforcement by its compressive action exerts a powerful effect to resist bulging and to prevent circumferential elongations so that the reinforcement (both lateral and longitudinal), enables it to withstand much greater deformations without cracking or checking than would be otherwise possible.

Let us review the action of the column as the load is applied with reference to the manner in which the potential energy of internal work is stored within the structure. As the load is gradually applied we have a certain elastic deformation and the internal work is the mean weight times the deformation. If yielding of the material occurs or scaling of the shell, a certain amount of energy is dissipated, equilibrium is destroyed, and new energy is developed by downward motion of the load thru increased deformation until a new condition of stability of equilibrium in established.

Scaling and cracking of the shell means a loss of potential energy stored and correspondingly larger deformations which are inadmissible in the practical structure. The addition of vertical steel prevents this checking and scaling and dissipation of energy because it provides a storage system of energy which is stable and in which the storage of energy by indirect stress can be depended upon.

Coaction between the hoops and the vertical steel reduces the deformation and hence the quantity of potential energy stored for a given load and correspondingly increases the efficiency of the structure as a load carrying mechanism.

9. Comparison of Test Data. Having pointed out in detail, the difference in action between these two kinds of columns, it is in order to compare test data. Column Table V, gives the reinforcement, percentage of steel, and test results by Withey* of a series of vertically reinforced and hooped columns with vertical steel flush with the ends and resting against bearing plates. These may be compared with the tests by Bach, and the tests at Phoenixville.

Taking up the comparison, first, with the series by Bach, the following point of difference is noticeable. The test results obtained by Withey can be substantially accounted for by considering the influence of the concrete and vertical steel alone. The test results by Bach, cannot be accounted for in this manner. Both series show that hooping adds toughness. Both indicate that larger vertical rods have more effect in raising the yield point than rods of too small diameter, which offer little resistance to lateral bending. The Bach tests, where the spiral pitch of is not too great, and where larger vertical rods are used with close spacing of spirals, are in good accordance with the Considère formula; while the tests by Withey are not in agreement but are in more close agreement with a different formula, which considers the manner in which the load is brought upon the steel, namely, by direct pressure instead of by bond shear as in usual operation of the column in practical building.

Comparing the tests by Withey with those at Phoenixville, the longitudinal reinforcement of the columns, in specimens, *J*, *N*, *P*, *O*, *R*, and *Q*, is substantially the same; while the hooping in the Phoenixville tests was much greater. The increase in ultimate strength in the Phoenixville test, for one percent of total steel, ranges from two to three times as much as in the tests by Withey. This

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*Bulletin of the University of Wisconsin, No. 466.

COL. TABLE V

Withey's Tests of Hooped Columns Longitudinal Steel Full Length of Shaft

Col. No.	Reinforcement: Round Vertical Rods No.	Reinforcement: Round Vertical Rods Size	Reinforcement: Spiral Size	Reinforcement: Spiral Pitch	Percent Reinforcement Longitudinal	Percent Reinforcement Lateral	Core area sq. in.	Age in days	Mix	Ultimate strength lbs/sq.in P/A	Stress at yield pt. in core lbs/sq.in P1/A	P1/P	Compressive strength of cylinders lbs/sq.in
1	2	3	4	5	6	7	8	9	10	11	12	13	14
C 1	0	0	¼" rd.	1"	0	2.00	78.5	62	1-2-4	4,660	2,720	0.58	2,280
C 2	0	0	¼" "	1"	0	2.00	78.5	62	1-2-4	4,390	2,330	0.53	2,190
C 3	0	0	¼" "	1"	0	2.00	78.5	64	1-2-4	3,660	1,950	0.53	2,180
C 4	0	0	¼" "	1"	0	2.00	78.5	65	1-2-4	3,410	2,330	0.68	2,150
D 1	9	⅝"	¼" "	1"	3.50	2.00	78.5	58	1-2-4	4,470	3,290	0.74	2,150
D 2	9	⅝"	¼" "	1"	3.50	2.00	78.5	60	1-2-4	4,200	3,480	0.83	2,130
D 3	9	⅝"	¼" "	1"	3.50	2.00	78.5	61	1-2-4	4,970	3,860	0.77	2,380
D 4	9	⅝"	¼" "	1"	3.50	2.00	78.5	62	1-2-4	5,360	3,670	0.68	2,350
H 1	0	0	No. 7	2"	0	0.50	78.5	57	1-2-3.5	2,330	1,950	0.84	2,040
H 2	0	0	No. 7	2"	0	0.50	78.5	57	1-2-3.5	2,140	1,760	0.82	*1,460
G 1	8	½"	No. 7	2"	2.00	0.50	78.5	49	1-2-3.5	3,320	2,710	0.82	2,100
G 2	8	½"	No. 7	2"	2.00	0.50	78.5	58	1-2-3.5	3,280	2,710	0.83	*1,420
I 1	8	11/16"	No. 7	2"	3.78	0.50	78.5	57	1-2-3.5	4,240	3,660	0.87	2,240
I 2	8	11/16'	No. 7	2"	3.78	0.50	78.5	57	1-2-3.5	4,080	3,280	0.80	2,120
J 1	8	⅞"	No. 7	2"	6.11	0.50	78.5	58	1-2-3.5	5,190	4,240	0.82	2,110
J 2	8	⅞"	No. 7	2"	6.11	0.50	78.5	58	1-2-3.5	5,050	4,240	0.84	2,000
L 1	0	0	No. 7	2"	0	0.50	78.5	57	1-2-3.5	2,680	1,370	0.51	1,780
L 2	0	0	No. 7	2"	0	0.50	78.5	58	1-2-3.5	2,600	1,370	0.53	1,760
K 1	8	½"	No 7	1"	2.00	1.00	78.5	57	1-2-3.5	4,050	2,710	0.67	2,100
K 2	8	½"	No. 7	1"	2.00	1.00	78.5	57	1-2-3.5	3,760	2,520	0.67	1,890
N 1	8	11/16"	No. 7	1"	3.78	1.00	78.5	57	1-2-3.5	4,050	3,470	0.86	1,880
N 2	8	11/16"	No. 7	1"	3.78	1.00	78.5	58	1-2-3.5	4,340	3,280	0.76	1,720
M 1	8	⅞"	No. 7	1"	6.11	1.00	78.5	57	1-2-3.5	4,790	3,860	0.81	1,660
M 2	8	⅞"	No. 7	1"	6.11	1.00	78.5	58	1-2-3.5	4,580	3,660	0.80	1,710
P 1	8	1"	No. 7	1"	8.00	1.00	78.5	57	1-2-4	6,760	5,560	0.82	2,380
P 2	8	1"	No. 7	1"	8.00	1.00	78.5	60	1-2-4	7,090	5,760	0.81	2,350
O 1	8	⅞"	¼" rd	1"	6.11	1.96	78.5	57	1-2-4	6,510	4,240	0.65	2,270
O 2	8	⅞"	¼" "	1"	6.11	1.96	78.5	57	1-2-4	6,650	4,620	0.70	2,690
R 1	8	1"	¼" "	1"	8.00	1.96	78.5	53	1-2-4	7,250	5,000	0.69	2,310
R 2	8	1"	¼" "	1"	8.00	1.96	78.5	53	1-2-4	6,680	5,380	0.80	2,470
Q 1	8	1⅛"	¼" "	1"	10.12	1.96	78.5	53	1-2-4	6,190	5,190	0.84	2,280
Q 2	8	1⅛"	¼" "	1"	10.12	1.96	78.5	53	1-2-4	7,990	6,340	0.79	2,330
W1	0	0	0	0	0	0	86.6	52	1-2-4	2,660			2,600
W2	0	0	0	0	0	0	86.6	52	1-2-4	2,660			2,400
W3	0	0	0	0	0	0	86.6	51	1-2-4	2,480			2,250

Note: All spirals 10" in diameter. Cols. C1-D4 are 120"-lg. All other cols. 102"-lg.

wide discrepancy, apparently can be accounted for only as above outlined, since the discrepancy is far too great for it to be possible to account for it on the ground of difference in the strength of the concrete. The concrete in the test of the Phoenixville specimens is undoubtedly closely comparable to the concrete of the specimens tested by Bach, and it is undoubtedly a somewhat better grade than the concrete in the specimens tested by Withey.

The conclusions that the authors draw from these tests are:

That it is bad practice to splice longitudinal reinforcing bars by bearing of one bar upon the other.

That it is better to lap bars of consecutive columns at the floor line in order that a natural adjustment may take place between the materials as the load is brought upon the concrete during erection.

That a column constructed by lapping the bars, so that the natural relations between the reinforcement and the concrete are conserved, that is, that indirect stress of the bond shear may be effective, substantially doubles the efficiency of the vertical steel; and hence this detail of reinforcing should be used. If this efficiency is to be counted upon, the diameter of the vertical bars should, for like reason, be limited preferably to 1⅛ inch diameter, or in special cases of heavy work to 1¼ inches to 1⅜ inch round bars.

That the hooping should not be spaced further apart than four inches for light pressures, and closer spacing should be used for heavy pressures.

That the spacing of the vertical bars should not exceed nine inches between centers, but if spaced further apart their efficiency should be considered to be reduced by fifty percent of the relative increase in spacing.

10. Formula for Columns, where the Load is Brought upon the Steel by Direct Bearing on Metal. Tests of this type of column show, up to the yield point, so-called, that the effect of the hooping is relatively small; that beyond the yield point the hooping is brought into play, giving the column a degree of toughness and ability to stand increased load without sudden failure, altho accompanied by considerable deformation. Neglecting the circumferential reinforcement and the concrete outside of it,

Let A = area of core occupied by concrete and longitudinal reinforcement having a steel ratio of p, so that

$A_s = pA$ = cross section of longitudinal steel.
$A_c = (1 - p)\ A$ = cross section of concrete core.
W = total load to yield point that column would carry without circumferential steel.
$e_1 = .00125''$ = average deformation per unit of length at yield point of column.

$W = (A_s E_s + A_c E_c)e_1$. Let $E_s/E_c = n$, then
$w = W/A = E_s e_1 [\ (1 - p)/n + p]$

is the load at yield point per unit of cross section of concrete.

Let $E_s = 30{,}000{,}000$, and $n = 15$; then
$w = 37{,}500\ [\ (1 - p)/15 + p'\ ] = 2500 + 35{,}000\ p$

The point here designated on the diagram of unit deformations and load is at intersection P of the tangents to the two slopes of the curve, Fig. 130.

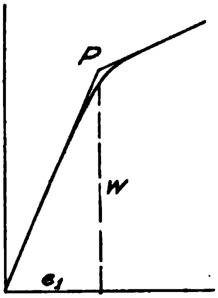

Fig. 130. Diagram Showing Elastic Curve of Column

Equation (1) is in good agreement with the series of tests made by Professor M. O. Withey, C. E., in 1910, at the University of Wisconsin,* with longitudinal reinforcement from 1 percent to 10 percent and circumferential reinforcement from 0.5 percent to 2 percent, given in Col. Table V. Professor Withey has written empirical formulas for several different percentages of circumferential reinforcement which differ slightly from this and from each other; but they do not refer precisely to the point P as above defined, but to a slightly smaller value of W. The point P does not actually fall on the test curve of unit deformations and loads, but is so related to it geometrically as to make its use advisable. The actual deformation at this load will be somewhat in excess of the assumed value of e_1.

On page 49, in Bulletin 466, University of Wisconsin, the following conclusions are drawn from the results of these tests regarding the behavior of columns with rigid metallic bearing for the longitudinal reinforcement built of concrete similar to those tested by Withey.

> 1. "Closely spaced spiral reinforcement will greatly increase the toughness and will considerably increase the ultimate strength of a concrete column, but will not materially affect the yield point. The ultimate strength under dead load will doubtless be somewhat less than the value obtained from a testing machine. Since spiral reinforcement should be employed principally as a factor of safety against sudden collapse, more than one percent does not appear to be necessary. On account of lack of stiffness in columns made from this grade of concrete and reinforced with spirals only, it seems necessary to use some longitudinal steel.
>
> 2. "Longitudinal steel in combination with such spiral reinforcement raises the yield point and ultimate strength of a column and increases its stiffness. As was shown in Bulletin No. 300 columns reinforced with longitudinal steel only are brittle and fail suddenly when the yield point of the steel is reached; but they are considerably stronger than plain concrete columns made from the same grade of concrete.
>
> 3. "From behavior under test of the columns reinforced with spirals and vertical steel and the results computed, it would seem that a static load equal to 30 to 40 percent of the yield point would be a safe working load."

---

*Bulletin No. 466.

These tests are conclusive as regards the deportment of the column without vertical steel. Spiral reinforcement gives a large increase in the ultimate strength and while providing safety against sudden collapse, does not increase the yield point at which the fireproofing commences to scale, and hence does not permit material increase in the working stress over that of the vertically reinforced column types with ordinary ties.

11. Working Stresses. (*a*) *The case of no direct metallic bearings.* In the vertically reinforced and hooped column, in which the hooping follows closely the percentage of the vertical steel recommended previously, namely, .35 to .45 of the volume of the verticals, and the verticals are not less than three quarters of an inch in diameter, and spaced approximately as recommended, the yield point of the column may be taken as eighty percent of the ultimate strength. This would give a yield point value, for a column to which the Considère formula is applicable of .80 x 1.5 = 1.2 times the crushing strength of the concrete times the core area, plus eighty percent of the yield point value of the steel in verticals and hoops.

Since a column of this character may be depended upon with certainty, if ordinary care is used in erecting the work, the factor of safety of 2.5 may be employed. This would give, in round numbers, 800 pounds per square inch for the concrete of the core, 12,000 pounds per inch on the verticals, and 16,000 pounds per square inch on the hooping, treated as imaginary verticals, for the working stress, the hooping being assumed to be a drawn wire, and possessing a higher yield point than the material of the verticals.

With a 1 : 1½ : 3 mix, twenty percent higher values may be assigned to the core.

(*b*) *Metallic bearings.* Where vertical reinforcement is used, and the load is brought on the vertical steel by bearing on metal, there is much less certainty of uniformity of joint action between the matrix and the reinforcement than where the longitudinal bars are lapped; and hence a somewhat higher margin of safety should be allowed. Taking this margin of safety as the reciprocal of 3¼, the writers would recommend a working stress not exceeding 700 pounds per square inch on the concrete plus 10,000 pounds per square inch on the vertical steel for this type of column with rigid end bearing, limiting the percentage of hooping to not less than one half of one percent nor less than twenty percent of the vertical steel, and the vertical steel limited to not more than eight percent of the column area. This factor of safety is based on the yield point value of the column.

12. Structural Columns Filled with Concrete. As steel is more rigid than concrete, the higher the percentage of the steel the less the assistance we may expect the concrete to be to the steel, and while the formulas given heretofore for the column in which the vertical steel is loaded in large part by direct bearing of metal on metal agrees with tests closely, it should be noted that the percentage of steel in these specimens was not high. Accordingly, it would seem conservative to reduce the allowance on concrete in cases where the percentage of steel exceeds eight percent of the combined area of the concrete and steel in proportion to the increase in steel above this percentage.

13. Concrete Columns Compared to Structural Steel. From the Phoenixville tests, it would seem that there is a higher degree of uniformity in the tested strength of reinforced concrete columns made with ordinary care and with well designed reinforcement than with average structural steel columns. The reason is that in the concrete column, we have a solid core. The larger the column the greater the strength, in strong contrast with some structural steel columns. In the structural steel columns, we have the uncertainty due to the form and make-up of the section. In the make-up of most small struts and columns constructed of steel and iron, we have a specific ratio of the area of the web and flanges consisting of channels, or in other forms, a complete circular or box section, such as the Phoenix column or the box made up of two channels and two plates. The semi-empirical formula for struts has been worked out for the radius of gyration of those forms of sections which are comparable in the case of laced channel struts to a certain distribution of metal between the web and the flange and a certain ratio of flange width to depth, which ratios, under standard sections, are comparable to the box and Phoenix sections. When there is a wide variation from these proportions, the formula based on the radius of gyration is inapplicable as is proved in the original design of the columns of the Quebec Bridge. In this case the area of the flange as compared to the web was about one tenth of this standard proportion between the area of the web and the channel, and there is no experimental data in existence covering the action of the latticed bars and secondary stress for such a combination.

In concrete work such uncertainties are eliminated by the uniform solid core. With ordinary care there can be no doubt about securing a solid casting, if the type of reinforcement selected is that recommended in this chapter, namely, one in which there is no obstruction whatever in the core to prevent the concrete from flowing freely and filling the same completely and so securing a solid casting. Improper handling of the concrete might lead to considerable reduction of the strength of the section, but the damage which might occur thru bad workmanship is if anything much less in concrete work than in steel construction and the confidence of the builder in the integrity of his concrete work should be correspondingly greater.

14. Wall Columns and Interior Columns in Skeleton Construction. Wall columns differ from the interior column in that the load comes to them from one side instead of equally or approximately equally from all directions as it does under a uniform load on the floors.

Whether the floor is beam and girder construction or flat slab construction supported directly upon columns in view of the monolithic connection between the floor and the columns certain eccentric loads or bending stresses are brought upon the wall columns to a greater extent than is the case with interior columns.

The amount of bending induced in the column will depend, first on the rigidity of the floor, and second, upon whether the resistance of the floor to deformation is furnished by beam action or circumferential slab action. If the resistance is by circumferential slab action, the effect upon the column is far less for the same deflection than it would be in case of beam action, because the slab tends to twist from all directions and so in a large measure the effect of the load in producing bending and deformation of the column is counter balanced by the loads coming to the column from the sides instead of being augmented by these loads so much as is the case in the beam action of one-way reinforcement.

Favoring Conditions. The walls of a building are made vertical and as the number of stories increases, the columns are reduced in diameter, but kept flush or vertical on the outer face. This produces an eccentric application of the column load from the upper stories upon the columns of the lower stories which in some large measure off-sets and holds in equilibrium the bending moment of the floor loads.

Little attention is usually paid to this difference in conditions. The wall column is commonly made approximately the same as the interior column except in the upper stories where it is possible to carry the wall columns up for the last three stories without reduction in size, making thereby a provision for the columns of these upper stories which would be most affected by the eccentric bending moment referred to.

Good practice would limit the dimensions of columns to sixteen inches as a minimum size in office buildings and in warehouse construction, eighteen inches as a minimum.

Provision of material in the beam to take the entire bending resistance figured as simply supported at the wall column and continuous over the interior columns will in no wise eliminate this bending stress in the column itself, and does not excuse failure to make such provision.

In fact if the beams are treated as beams fixed at the inner ends and free at the outer end, and the five rod type of reinforcement used in the Turner beam is adopted, the positive moment provided for near mid span is $W L/14.4$ nearly, or $.07W L$, while the maximum positive moment in the beam freely supported and continuous over a number of spans in the end span requires a coefficient of from 0.07 for two spans to .08 for three spans, and .077 or .78 for a greater number, so that in this type of design there is nearly sufficient provision without considering the stiffness of the wall column. Nevertheless, in the view of the authors the provision recommended should be followed in practice.

Bending from Wind Pressure. In view of the monolithic character of reinforced concrete construction, wind has much less effect on a structure of this kind than is the case with the steel frame, since each floor forms a monolith of enormous lateral rigidity causing columns all to act in perfect unison. Where the building is narrow the columns should be fully spliced at the floor level which doubles the resistance of the column to flexure. Where the buildings are broad and only six to eight stories in height, provision of this character is unnecessary except in wall columns which should be well spliced.

15. Temperature Effect on Columns. Changes in temperature cause expansion and contraction of the concrete floor. This is taken up by bending in the column and the in and out motion of the walls where they are above the ground line. There is, however, a tendency for the basement walls to crack every thirty or forty feet because of the restraint of the surrounding earth. There is a further tendency for brick walls to crack where there is expansion if brick piers are used in place of well reinforced concrete piers in a long building. In a building over 300 feet in length, no dependence should be placed upon either a plain brick or a plain concrete pier without reinforcement to withstand temperature stresses of expansion and contraction of the floors since unsightly cracks and checks will very likely occur from this cause. In a building over 300 feet in length either a full, well reinforced concrete skeleton should be provided or if bearing walls are used the building should be cut at 250 foot intervals so that it will allow expansion and contraction without damage to the brick work or undue strain on the concrete of the floors and columns.

16. Economic Column Design. With reinforcing steel in the column figured at 2½ cents a pound and concrete at $6.00 per cubic yard, we may estimate the proportion of steel and concrete needed to obtain the most economic support from the standpoint of cost. Since a cubic foot of concrete weights 150 pounds and a cubic foot of steel 485 pounds, a cube of concrete weighing one pound would have a volume 3.2 times as great as a cube of steel of the same weight, and have a face nearly one and one-half times as great. On this basis, for equal cost in carrying load, $f_s/f_c = 37$, but in a fairly rich concrete $n = 10$ to 15, so that a load can be carried with less cost on a concrete pier than upon one of reinforced concrete, but in carrying the load in this manner, the diameter of the columns are greater, which is objectionable in that they occupy valuable room. Unbalanced bids are sometimes received on this basis. One puts in a bid using a column of a small diameter as specified, thus using more steel, but saving space; another makes a column larger by two or three inches and uses less steel and puts in a lower price. The owner is frequently persuaded to accept the large columns without giving the regular bidder an opportunity to revise his bid on the same basis.

It should be noted that the efficiency of the hooping varies as the diameter of the core, while the core area varies as the square of the diameter; hence a small increase in the diameter of the core permits a large decrease in the amount of hooping and vertical steel. Hooping costing 1.6 times as much as the vertical bars in a column in place has an efficiency 2.4 times that of the same cost of metal in vertical steel. On the other hand, certain relative proportions between the amount of the vertical steel and hooping are necessary to secure the best results, and these proportions should be adhered to if the formulas recommended are to be applied.

17. Column Formulas Dependent on Polar and Transverse Moments of Inertia. What the correct formula should be for compression members or columns of structural steel has for years been a subject of theoretical discussion and controversy. The formulas that have been adopted have, without exception, undertaken to determine what the working stress should be from the moments of inertia of the section that opposes flexural deformation, whereas most tests of structural steel columns clearly disclose failure by combined torsion and bending. Therefore, an accurate formula for a structural steel compression member must make its strength depend on polar as well as transverse moments of inertia. But the polar moment of inertia in many forms of steel columns reaches a very small value and the members are weaker than any computation which takes into consideration only their resistance to flexure in various directions and disregards their lack of stiffness under torsional deformation, indicates.

Because in the concrete column the section is solid and the polar moment of inertia is relatively so great that failure actually occurs by crushing and shear or flexure, the combination of polar moment with transverse moment of inertia has in this case no appreciable effect to diminish the resistance of the column. Accordingly, since plain crushing and transverse bending is by far the simpler phenomenon the uncertainty as to the safety of reinforced concrete columns when calculated by scientific formulas is far less than that involved in the use of the usual formulas to calculate the ultimate strength of structural steel members as commonly built.

18. Conclusion. In the preceding discussion of reinforced columns the authors have advocated higher working values for columns when vertically reinforced and hooped in proper proportions than is permitted by many building code regulations. On the other hand, nearly all such regulations allow working stresses upon the vertically reinforced column with ties higher than the authors consider safe in practice. The resistance of the hooping depends on the lateral expansion of the column under longitudinal compression.

In the column tests by F. P. McKibben and A. S. Merrill, Proc. Am. Conc. Inst., 1916, pp. 200-240, careful measurements were made to determine the lateral expansion of the column but as was the case in the beam tests by the Bureau of Standards no effort was made to ascertain the initial state of stress in the column produced during the chemical action of hardening.

Cracking or checking of the fireproofing occurs at low unit stresses where hooping only is used but with proper proportions of vertical steel the fireproofing remains intact under high intensities of load.

Because measurements under initial loading of columns show small hoop tension under compression not exceeding working stresses, the argument is advanced that the hooping provided to resist lateral expansion cannot be counted upon for elastic resistance of high stresses. The initial state of stress, however, is modified by repeated loading and continued use just as certainly as in the case of beams in which under initial loading the measured steel stress accounts for but a small part of the load moment. From this latter fact no one would conclude that the steel should not be counted upon to resist the load moment. Neither should similar data lead to the conclusion that hooping is not to be counted upon to resist high stresses in the column. Column reinforcement as well as beam reinforcement is largely affected by the state of stress induced during the hardening of the concrete, a matter yet to be fully investigated to complete our understanding of the co-operation of hooping and vertical reinforcement.

A thorough comparison of the working stresses allowed by different building codes throughout the United States by P. P. Furber, was published in Proc. Am. Conc. Inst., 1916, pp. 181-199 from which the following quotation with which the authors are in full accord is made.

> "Supposedly it is well known to structural engineers that columns vertically reinforced and hooped with sufficient spiral reinforcement of close pitch are much to be preferred to columns with vertical rods only, even in case ties are used, as they were not in the Edison buildings. In spite of this knowledge, columns without hooping are being built extensively throughout the country, although it is common practice to use ties or stays of small diameter rods or wire for lateral reinforcement, these being spaced about a foot apart as a rule. The reason for this inferior construction is found in the lower cost of that type as compared to the hooped column, when designed according to most building ordinances.
>
> Many building ordinances place a premium on this dangerous construction by allowing higher relative stresses on the concrete of such columns than on spirally reinforced columns, when we consider the increase in ultimate strength of the latter. Some have gone so far as to permit as great unit pressure on the concrete of a column without hooping as on a well designed hooped column. In other words, the concrete of tied columns is stressed beyond what should be permissible values, and the hooped column, instead of being given the benefit of its increased toughness and reliability, is actually put at a disadvantage because the allowed stresses are lower than tests have shown to be safe.
>
> The results of such building laws are obvious. Designers will use the cheaper type, except in some cases where it may be possible to control the design of the work with regard to actual strength

secured in the structure rather than with a view to meeting the requirements of some building department. It may be said that a conscientious engineer will always work on the former principle, but this is not so, and will hardly be possible where there is an ordinance limitation. The engineer has no opportunity to use his best judgment. The department officials and framers of the codes have exercised that function so that all one can do is to follow the rules and get the cheapest thing possible.

The discriminating engineer would rather be responsible for the construction of a spiral column—even with unit stresses considerably above those allowed by many building ordinances—than for columns without hooping, under the usual values for unit pressure. The difference between intelligent and inferior design may be judged better by the type of members and details selected than by the figured unit stresses.

Most of the failures of concrete structures have occurred in buildings designed so that the figured unit stresses were not too high when judged by ordinance requirements, and the failures have been due to failures of the type of member rather than of the material. A large number can be blamed on the use of columns without hooping. On the other hand, the writer at least has never heard of the failure of a hooped column in a building. The continued service of hooped columns under pressures, which, according to some building laws, would be considered dangerous, leads one to believe that something is wrong with the building laws, particularly when reliable column tests indicate that the stresses allowed are conservative.

Some may argue that the fact that a building stands up is no sign that it was well designed. This is quite true, and applies with force to many buildings in which rodded columns have been used. In considering designs in the true sense, that is, with reference to the type, it should be the duty of the engineer to avoid members that are subject to sudden failure."

CHAPTER XII

UNBALANCED MOMENTS IN MONOLITHIC FLOORS AND COLUMNS

Introductory. A monolithic reinforced concrete slab floor with its column supports is an indeterminate structure, the deformations due to bending under unbalanced loads being distributed between the columns and the slab as shown in the accompanying figure.

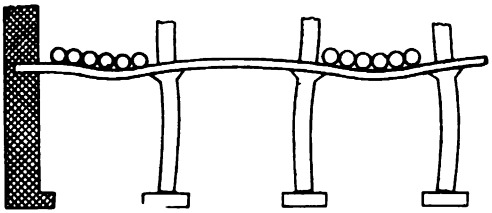

Fig. 132

Thus the panel of the slab at the left in Fig. 132 built into the wall is bent downward by the load on that panel. The wall furnishes restraint at that end of the panel as does the column above and below at the right. The column, restrained at the footing below, is bent in a reverse curve while the unloaded panel is bent upward. The next panel which is loaded is bent downward. The bending of the column in each case is toward the unbalanced load. Like bending occurs where there are beams built as a monolith with the columns in place of the slabs.

Since reinforced concrete buildings are indeterminate structures the resistance to applied moments arising from unbalanced loads must be sub-divided between the slab itself and the supporting columns according to their relative rigidities, but the equations for determining these resistances on the principle of least work are too complicated for practical use. A more simple method, that is sufficiently exact for practical purposes may be developed as follows:

The method depends in the first place upon an investigation of the properties of continuous beams which are also indeterminate structures, secondly upon the investigation of the action of unbalanced loads independently of that of the balanced loads. Since flat slab floors like other engineering structures are regarded as elastic, the stresses due to separate kinds of loading such as balanced and unbalanced loads may be computed separately, while the total actual deformations will be the algebraic sum of the component deformations arising from these separate loads.

This Chapter will treat the deformations and stresses arising from unbalanced loads alone, assuming that a separate computation of balanced loads has been made.

2. Theory of the Uniform Continuous Beam. Consider the action of unbalanced loads upon a beam AB Fig. 132 whose weight is to be neglected because its weight constitutes part of the balanced loads. Assume further that the Span $OA=L$ is free from unbalanced loads, but that the other spans may have unbalanced loads applied to them of arbitrary position and magnitude and such as to cause the span OA to fulfill the conditions that will be required of it at the supports O and A, or elsewhere. The first of the conditions that will be required is that the beam shall be bent in such a manner as to tip thru a certain assigned small angle v at O, where it is integrally joined either to a single column such as OD below AB, or to two columns OD and CC, one below and the other above AB.

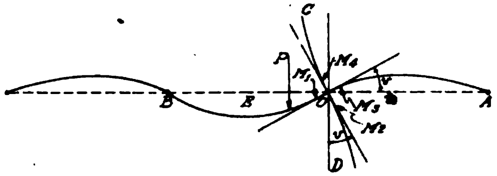

Fig. 132

In reality the foregoing requirement embraces two conditions at once, viz., one is the pinning of the beam to the support O regardless of its direction at that point, and the other is restraining it at the fixed slope v at O. Let a second requirement be that the beam be also pinned to the support A either with or without restraint as to direction, as will be stated later. All these conditions and restraints will be brought about by loads such as P situated in spans at the right and left of AO and by integral connections with columns, etc. Let the tipping of the beam at O thru the angle v be regarded as produced by the application of the moment M_1 in the beam just at the left of O, which at the same time will cause the column or columns integral with the beam at O to become inclined to the vertical at the same angle v by which the beam is made to incline from the horizontal at O.

This applied moment, M_1 may be taken as due to unbalanced loads P at the left of O and reaction at O. The moment M_1 is held in equilibrium by the resisting moments M_2 and M_4 in the columns below and above O, assisted by M_3 the resisting moment in the part of the beam immediately at the right of O, Hence,

$$M_1 = M_2 + M_3 + M_4 \qquad (1)$$

is the equation of equilibrium of the applied and resisting moment called into play at O by the unbalanced load. It should be specially noted that since no tipping of the beam whatever can be produced at O by any balanced loads, v depends solely upon the unbalanced part of the total load, and will be unchanged whether a balanced load is present or absent, because in an elastic structure the independent deformations are merely superposed on each other. The same is true at every support and at mid span between supports, but at all other points the inclination will depend both on the balanced and the unbalanced loads.

Apply to the span OA the known general theorem that the sum total of the numerical values of the positive bending moment at mid span added to one half the sum of the numerical values of the negative moments at the ends is equal to the constant $\frac{1}{8}wL^2$ when w is the uniformly distributed load per unit of span. In this case $w=0$, hence

$$M_o - \tfrac{1}{2}(M_3 + M_A) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which M_o is the positive moment at mid span, M_3 and M_A are the negative moments at O and A respectively. These moments are not due to the balanced loads in any way whatever, but are the parts of the total moments at these points due to the unbalanced loads alone. The total moments will be obtained by adding to these the moments due to the balanced loads.

In order to investigate the moments, the slope, and the deflections occurring in the span OA which are due to unbalanced loads in spans other than OA, let M be the bending moment at any cross section of the span OA situated at a distance x from O, then

$$M = EI\frac{dy^2}{dx^2} = Rx + M_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

in which R is the reaction at O, or the shear in a cross section immediately at the right of O.

Integrating (3) we have

$$EI\frac{dy}{dx} = \tfrac{1}{2}Rx^2 + M_3x + C \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

but at O, $x=0$, $\dfrac{dy}{dx} = v$, or $C = EIv$ $\ldots\ldots\ldots\ldots\ldots\ldots$ (5)

Substituting this in (4) and integrating, we have

$$EIy = Rx^3/6 + \tfrac{1}{2}M_3x^2 + EIvx + C' \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

At O, $x=0$, $y=0$, hence $C'=0$

Where $x=L$, $y=0$, hence

$$\tfrac{1}{2}RL^2 + \tfrac{3}{2}M_3L + 3EIv = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

These equations express relations that hold true for all degrees of restraint at A such as would be produced either by fixing the beam at A at any given inclination to OA, or by applying any assigned bending moment to the beam at A or at any other given point of OA. This is evident because with (7) given, and either the slope $\frac{dy}{dx}$ or the moment M given at some known point of the span distant x from O, the determination of M_3 in terms of the known inclination v would follow from eliminating R between (7) and either (3) or (4).

1st For example at $x=L$, let $\dfrac{dy}{dx}=-v$, then

by (4) $-EIv=\frac{1}{2}RL^2+M_3L+EIv$

Hence by (7) and this equation we have

$$M_3=-2EIv/L \qquad (8)$$

In this case $R=0$, and M has the constant value M_3 at all values of x and hence the beam is bent in the arc of a circle, and the sum of the bending moments at O and A is $M_3+M_4=4EIv/L$. These conditions could be brought about most readily by equal loads symmetrically situated in the spans at the right and left of the span OA.

2nd At $x=L$, let $M=0$, in which case the beam terminates at A where it is simply pinned to the support, so that it is free to assume any direction at that point compatible with the slope v at O. Hence by (3)

$$0=RL+M_3, \text{ or } R=-M_3/L$$

Substitute this in (7), hence

$$M_3=-RL=-3EIv/L \qquad (9)$$

and from (4) we find the slope at A is $-\frac{1}{2}v$.

3rd At $x=L$ let $\frac{dy}{dx}=0$, in which case the beam is built in or fixed horizontally at A. Then by (4) $0=\frac{1}{2}RL^2+M_3L+EIv$ which combined with (7) gives for this case

$$M_3=-4EIv/L \qquad (10)$$

$$R=6EIv/L^2=-\tfrac{3}{2}M_3/L$$

While by (3) the bending moment at A is

$$M_A=RL+M_3=-\tfrac{1}{2}M_3$$

In this case by (3) we find that $M=0$ at $x=-M_3/R=\frac{2}{3}L$. But by (9) and (10) M_3 is in this case the same as in a beam simply pinned at a point distant $\frac{3}{4}L$ from O, because if the length from O to A in (8) be made equal to $\frac{3}{4}L$ we get the value of M_3 given in (9).

Furthermore we find in this case by (6) that at $x=2L/3$ we have $y=2vL/27$, which is the elevation of the point where $M=0$ above the horizontal line OA. By (3) the positive bending moment at A is $M_A=2EIv/L$.

Other cases may be treated in a similar manner with results which have been given for convenience of comparison in the following table in which either the moment at A where $x=L$, or the slope at that point may be given. With either one given the other properties of this span OA of the beam become completely known as already explained, and we have values such as are given in the table.

TABLE OF END MOMENTS AND SLOPES, ETC., OF AN UNLOADED SPAN OA WHEN THE END SLOPE $\left(\frac{dy}{dx}\right)_{(x=o)}=v$ IS GIVEN, AND EITHER $\left(\frac{dy}{dx}\right)_{(x=L)}$ or $M_{(x=L)}$ IS ALSO GIVEN.

	$M_{(x=o)}$	$M_{(x=L)}$	$M_{(x=\frac{1}{2}L)}$	R	$\frac{dy}{dx}_{(x=L)}$	$\frac{dy}{dx}_{(M=o)}$	$X_{(m=o)}$
1.....	$-2EIv/L$	$-2EIv/L$	$-2EIv/L$	0		0	
2.....	$-3EIv/L$	0	$-3/2EIv/L$	$3EIv/L^2$	$-\frac{1}{2}v$	$-\frac{1}{2}v$	L
3.....	$-4EIv/L$	$+2EIv/L$	$-EIv/L$	$6EIv/L^2$	0	$-\frac{1}{3}v$	$\frac{2}{3}L$
4.....	$-5EIv/L$	$+4EIv/L$	$-\frac{1}{2}EIv/L$	$9EIv/L^2$	$+\frac{1}{2}v$	$-\frac{7}{8}v$	$\frac{5}{9}L$
5.....	$-6EIv/L$	$+6EIv/L$	0	$12EIv/L^2$	$+v$	$-\frac{1}{2}v$	$\frac{1}{2}L$

The usual conditions fulfilled by a continuous beam which has an unbalanced load on a single span only, such as OB, are represented by the values in cases 2 or 3, or some intermediate values dependent upon some degree of restraint at A between perfect freedom of motion, *i. e.* $M_A=0$, and perfect horizontal restraint, *i. e.* $\left(\frac{dy}{dx}\right)=0$ at the point where $x=L$.

In case the continuous beam is integral at A with columns, or is restrained by other parts of the structure, that will cause the beam to approach more or less closely to case 2.

The same considerations hold respecting the columns OD and OC below and above O as apply to the span OA at the right, viz.: the ends of the columns away from O will in general have a degree of restraint that is nearly perfect either by their footings or junctions with other parts of the structure, and they may be assumed without appreciable error to require the same coefficient b intermediate between —3 and —4 in expressing their resisting moments at O, as does OA, *i. e* where $x=0$.

$$M=b\,E\,I\,v/L \qquad \text{(11)}$$

is an equation which applies to either of the resisting members at O in case suitable values be ascribed to I and L in each member, while common values of b and v apply to all of them.

Now using the subscripts previously employed to distinguish the several members that meet at O, viz., 1 for the beam at the left, 3 for that at the right, 2 for the column below and 4 for that above AB, we hence have by (1) and (11)

$$\frac{M_3}{M_1}=\frac{I_3/L_3}{I_2/L_2+I_3/L_3+I_4/L_4} \qquad \text{(12)}$$

with similar expressions for M_2/M_1 and M_4/M_1. These expressions will show how the applied moment M_1 is subdivided among the resisting moments of the columns and span OA in case the structure is of homogeneous materials. In case of composite materials the formula will not be quite so simple because it will not be possible to cancel a common value of E from both numerator and denominator, as has been done in (12).

In case of a deck slab the column above O is absent, and in that case $I_4=0$ in (12). For any given or assumed value of the applied moment M_1 the effect of the absence of OC from the structure is to increase both M_2 and M_3, since the removal of OC causes a loss of rigidity which obliges the remaining parts to furnish the entire resistance instead of merely a fraction of it. For example, were $I_2/L_2=I_3/L_3=I_4/L_4$ in (12) then would $M_2=M_3=M_4=\frac{1}{3}M_1$. But were $I_4=0$ and $I_2/L_2=I_3/L_3$ then $M_2=M_3=\frac{1}{2}M_1$. In like manner were the beam to end at O, and not extend into the span OA, then $I_3=0$ and the entire applied moment M_1 must be resisted by the columns with which it is integral at O, so that M_2 and M_4 would be much increased thereby.

The case that was considered in what has preceded is one in which the same degree of restraint is assumed to exist in both the columns and in the beam at the extremeties designated by A. In case, however, where the beam terminates at A and merely rests upon a support at that point while the ends of the columns distant from O are built in so as to remain nearly fixed in direction it is evident from comparison of (9) and (10) that we must replace I_3 in (12) and the similar equations by $4I_3/3$, a reduction of rigidity of OA that will make M_3 less, and both M_2 and M_4 larger, than in the previous case.

3. The Moments and Deflections in a Single Uniformly Loaded Span of a Continuous Beam. Let the span OB be assumed to be of length equal to $OA = L$. Let it be uniformly loaded with an unbalanced load of w per linear unit, so that it constitutes the middle one of three equal spans on four symmetrical supports, and otherwise the continuous beam is supposed to be symmetrical in all particulars to the right and left of the center E of the span OB, there being no loads upon the end spans.

The differential equation of the neutral axis of OB in case the origin is taken at O and x is measured to the left is

$$EI\frac{d^2y}{dx^2} = M_1 + \tfrac{1}{2}\,w\,L\,x - \tfrac{1}{2}\,w\,x^2$$

in which M_1 as previously is the bending moment just at the left of O. Integrating we have

$$EI\frac{dy}{dx} = M_1\,x + \tfrac{1}{4}wL\,x^2 — wx^3/6 + C$$

At O, $x=0$, and $\frac{dy}{dx} = v$, hence $C = EIv$

At E, $x = \frac{1}{2}L$, and $\frac{dy}{dx} = 0$, hence

$$0 = \tfrac{1}{2}M_1L + \frac{wL^3}{24} + EIv$$

Hence by eliminating v, we have

$$EI\frac{dy}{dx} = M_1\,x - \tfrac{1}{2}\,M_1L + \frac{wLx^2}{4} - \frac{wx^3}{6} - \frac{wL^3}{24}$$

or $$EIy = \tfrac{1}{2}M_1x^2 - \tfrac{1}{2}\,M_1Lx + \frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24} + C'$$

At O, $x=0$, and $y=0$, hence $C'=0$.

At E, $x = \frac{1}{2}L$, and $y = y_c$, hence $EIy_c = -\frac{1}{8}M_1L^2 - \frac{5wL^4}{384}$

From this equation the central deflection y_c may be calculated for the several different values of the applied unbalanced moment M_1 that occur by reason of different assumed degrees of restraint at the supports, etc. These different values of M_1 as well as those of M_c may be determined when the restraints at the supports are given, for the value of M_1 will be found by the theorem of three moments and then M_c from the equation

$$M_c + M_1 = \tfrac{1}{8}wL^2$$

The results of the application of the theorem of three moments and our equation for central deflections to three pertinent cases is as follows:

Case 1. Beam fixed horizontally at both *B* and *O*

$$M_1 = -\frac{wL^2}{12}, \quad M_c = \frac{wL^2}{24}, \quad EIy_c = -\frac{wL^4}{384} = -0.0026\,wL^4.$$

Case 2. Beam fixed horizontally at end support *A* and the other end support but free to tip at *B* and *O*.

$$M_1 = -\frac{wL^2}{18}, \quad M_c = \frac{wL^2}{14.4}, \quad EIy_c = -\frac{7wL^4}{1152} = -0.0061\,wL^4$$

Case 3. Beam pinned but free to tip at all four supports.

$$M_1 = -\frac{wL^2}{20}, \quad M_c = \frac{3wL^2}{40}, \quad EIy_c = -\frac{13wL^4}{1920} = 0.0068\,wL^4.$$

It is to be noted that the deflection in the second case is two and one third times that in the first case by reason of the reduced rigidity, and by reason of the further reduction of the rigidity in the third case the deflection is increased to 2.6 times that in the first case. We may say then that lack of firm restraint by the columns at the ends of the span *BO* will in general more than double the central deflection. Such lack of firm restraint may be due either to lack of integral connection between columns and beam or to the small moment of inertia in the columns themselves as compared with that of the beam.

4 Unbalanced Loads on a Floor Resting on a Rectangular System of Continuous Girders Integral with its Supporting Columns. Suppose continuous beams both lengthwise and crosswise of the floor between columns, then in addition to the resistances previously considered as acting at O in a continuous beam, any applied moment M_1 will be resisted by twisting moments M_5 and M_6 in the cross beams extending horizontally away from *O* in each direction at right angles to *AB*. Each of these moments of resistance will be of the form

$$M = -vFJ/L \quad \text{..................................} (13)$$

in which F = modulus of elasticity for shearing, J = polar moment of inertia, and L = effective length of cross beam from *O* to the fixed end, while the angle of twist $= v$, the inclination at *O* of the columns and longitudinal beams due to the applied moment M_1.

We shall then have by reasoning like that previously employed respecting (10)

$$M_1 = M_2 + M_3 + M_4 + M_5 + M_6 \text{..................} \quad (14)$$

$$\frac{M_5}{M_1} = \frac{FJ_5/4EL_5}{I_2/L_2 + I_3/L_3 + I_4/L_4 + (J_5/L_5 + J_6/L_6)F/4E} \quad (15)$$

with a similar expression for M_6/M_1.

The expressions for M_3/M_1, M_2/M_1 and M_4/M_1, will have the same numerators as in (12) etc., and the same denomenator as in (15).

In case any of the members here considered are absent, the values of the moments of inertia of those members are to be placed equal to zero in all these expressions, just as has been done in the previous part of this treatment. It is to be noticed, since moments are directed quantities, that the transverse moments are independent of the lengthwise moments and that the twisting moment in any beam is independent of its bending moment.

The effect of the twisting resistance of cross beams is to increase the rigidity of the structure and to decrease the bending moments at *O* in the columns and in the span *OA*.

It should be noticed that no twisting deformation may be produced when both ends of a transverse beam are twisted thru the same angle, as is the case when a bay of panels side by side are each loaded with equal unbalanced loads. The case treated in (14) and (15) is that of an unbalanced twist. A live load placed upon a single panel of such a floor with *O* at the one corner of the panel would not cause twists in any of the four beams at the edges of that panel, but would cause twists in all the other eight beams radiating from the corners in case the floor is assumed to be merely supported upon them. In case the beams are integral with the floor the tipping of the floor at the panel edges will twist the beams,

5. Flat Slab Floors. The foregoing investigation may be applied with suitable modifications to the unbalanced moments in a flat slab floor integral with a rectangular array of supporting columns; as for example where there is no unbalanced load to the right of an entire transverse row of columns such as *CD* one behind another across the entire width of the floor while the unbalanced load at the left of this row is so situated as to make the applied bending moments equal to each other in the several panels of the bay just at the left of the row *CD*. In such a case, it is evident that the entire floor or either of the bays or strips of panels at right angles to this crosswise row of columns *CD* constitutes a continuous beam and I_2

may be taken to designate either the sum of the moments of inertia of all the columns such as OD in the row below the floor, and I_4 the sum of those above, in the entire width of the floor, while I_3 is the moment of inertia of a section extending the entire width of the floor just at the right of this double row of columns, or otherwise, I_2 and I_4 may be used to designate the moments of inertia of the single columns respectively of this double row, and I_3 that of a single panel width of the floor. This latter meaning will be employed in this chapter so that I_2 and I_4 will each refer to a single column below or above the slab, and I_3 to a single panel width of the floor. Equation (12) may then be applied among other cases to the unbalanced moments, per panel just at the right of a single transverse bay of panels extending across the entire width of the floor when the bay is uniformly loaded. Designate these unbalanced moments per panel as longitudinal. The transverse moments along the uniformly loaded transverse bay considered as a beam are balanced and are greater than the lateral unbalanced moments that would occur after the removal of the uniform load from any of the panels of the transverse bay.

When every panel in each transverse bay or aisle across the width of the floor parallel to the row of columns CD is loaded alike, altho the several transverse bays may not be loaded in the same manner, no unbalanced twists are produced about any line across the floor, but if the panels are not loaded alike in any given transverse bay then such twisting moments are called into play within the bay and adjacent bays as to reduce to some extent the longitudinal moments. In any such case equations (15) are to be applied and in them J_5/L_5 and J_6/L_6 refer to panels of the bay from which loads have been removed. In case of partial removal of panel loads such fractions only of the just mentioned quantities are to be introduced as are equal to the fractions of the panel loads removed, so that where none is removed these quantities do not appear.

There is one modification, however, that must be made in applying equations such as (15) to a continuous flat slab floor by reason of the fact that these equations as originally obtained referred to twisting moments in beams which were unrestrained at any points between their ends, so that the twisting moment M_5 or M_6 which was applied at the end O was transmitted laterally to the other end undiminished by any intermediate restraint or bending moments acting upon its sides. But this is not the case in a continuous floor slab, because any twist develops bending resistances near O

across the sides of the crosswise strip which undergoes twisting, such resistances being distributed along those sides for some indefinite distance dependent upon the rigidity of the slab. The practical effect of this is to increase the resistance of the slab to bending. It thus appears that the final result in the case of the flat slab may be expressed correctly by (12) and similar equations, without the introduction of any polar moments of inertia J of the strip, provided the value of the moment of inertia I_3 be somewhat increased to include a section of somewhat more than a panel in width. For purposes of design no such modification need be made, because larger values of the resisting moment M_2 and M_3 and M_4 will be found, when the floor is so loaded that no twisting moments exist, than will be found after the removal of any part of the load. The greatest unbalanced bending moments M_2 and M_3 and M_4 occur under the same circumstances of longitudinal loading as in case of continuous beams.

It is therefore unimportant to consider the twisting moments that may occur in interior panels, excepting those that occur regularly in each panel with balanced as well as with unbalanced loads.

The manner in which that total moment along the edge of the panel is resisted depends upon the relative rigidities of the cross section at that edge. If no reinforcing steel is placed in the top of the slab near mid span, as is frequently the case, the position of the applied moment for this part of the edge produces twisting moments that transfer that portion of the applied moments toward the columns to points where the resistance of the side belts over the column heads affords the required resistance.

Deck floor slabs are also included as a special case of the foregoing theory by making $I_4 = 0$. In deck slabs the loss of rigidity due to absence of columns above the slab brings more of the applied moments to bear upon the remaining members in the same manner as was noted in the case of beams.

It is to be specially noted that in (12) and similar formulas the column lengths have a very important influence such that for spans of 20 feet or less and for column lengths of 10 feet or thereabouts the unbalanced moments in the lower floors of high buildings, where I_2 and I_4 are necessarily large, are largely resisted by the columns, while for deck floors expecially those upon tall columns the columns furnish comparatively little resistance to the unbalanced applied moment which last is resisted almost entirely by the slab itself with little assistance from the columns. As before stated, the unbalanced moments here considered are those arising from unbalanced loads, and are in addition to those arising from symmetrical loads or those uniformly distributed on all the panels.

6. Wall Panels. Any loading, whatever, even the dead weight of the floor itself, is an unbalanced load in this case and produces an applied moment at the exterior wall which can be resisted by the slab only by cooperation with the wall columns or the restraint of the walls themselves where the slab is built integrally therewith. Where the wall columns are sufficiently rigid the wall panel might be practically restrained at the wall column but in the construction of the wall panel the building of the slab into the wall or its integral connection with the wall beam renders its deflection between columns along the wall negligible. It therefore tends to induce a different curvature in wall panels from that which occurs in interior panels.

Under the principles demonstrated in Chapter VIII, Section 9 the advantageous operation of radial and circular stresses depends upon double curvature. Now the straightening out or the elimination of the side deflections between columns along the wall tends to induce cylindrical curvature unless the difference in rigidity on the four sides of the panel is so equalized that equivalent relations are maintained. In the older type of Mushroom construction, the half head of the wall column and its radial bent out column rods are depended upon to shorten the span between the wall column and the next interior column and maintain sufficient rigidity between the wall columns and interior columns so that the operation of the slab would be practically the same as in interior panels. The elimination of the rigid frame work of the spider column head in flat slab designs often met with has produced a special weakness in the wall panels of such floors and a tendency to deform in cylindrical rather than dish shaped curvature.

The most scientific manner of treating wall panels is to increase the rigidity of the sides of the panel normal to the wall so that their deflection will be a mean between the deflection of the edge between the interior columns and that on the wall side, which is of negligible amount. This is accomplished to a large extent in the Spiral System by the diagonal bands which are supplemented by additional rods in the wall panels and by additional direct rods thereby increasing the total metal from twenty to twenty-five percent in the sides of the panel normal to the walls over that of the interior side parallel to the wall. By this means wall panels develop substantially the strength of interior panels, when the rigidity of the wall columns has been so proportioned that adequate restraint is secured.

7. Moments of Inertia of Flat Slab Floors and Supporting Columns. In attempting to calculate the numerical values of I, etc. there are several particulars in which the moments of inertia of flat slabs are not entirely uniform thruout the spans from column to column, The largest deviation from uniformity is due to the customary large increase of steel (and frequently of concrete also) at and around the column heads. It is usual to take account of this by making a suitable change in the length of the span by assuming the effective length of the span to be reduced to some 85 percent perhaps of the actual distance between column centers. Now the column length suffers a similar reduction by reason of its capital. The effect of these two simultaneous reductions upon (12) and similar equations is largely to neutralize each other, consequently this effect may be neglected without material error.

So far as the moment of inertia of the rest of the panel except the column heads is concerned it will be possible to assign an outside value to it large enough to include all the resistance it exerts by assuming that every part of the area of the panel is entirely covered on its tension side by two layers of crossed rods spaced at the same average distance between centers as those in the several belts of slab steel. If these crossed rods are sufficiently near each other to make a fine-grained reinforcement, as they are when the distance from center to center of rods is somewhat less than the thickness of the slab, then its action in conjunction with the concrete will be to afford a tensile resistance and moment of inertia regardless of the directions of the belts practically the same as that of a continuous sheet of metal of the same weight as the total weight of the crossed rods.

The evidence of the truth of the statement that a fine-grained reinforcement of belts of crossed rods in contact with each other affords approximately the same resistance to tension in all directions as a continuous sheet of metal of the same total weight is found in the recorded results of extensometer measurements of load tests, but these measurements show a still further increase in the resisting moments of the steel due to the fact that the proportion of the total bond shear that is held in equilibrium laterally by mutual coaction between the reinforcing rods is larger than that held in equilibrium otherwise.

It is to be noted in this statement that the action of the concrete by its coaction with the reinforcement is explicitly allowed for, and no direct tensile stress in it parallel with the steel need be considered. The state of stress here under consideration presupposes a maximum unit tensile stress in the steel of 5000 lbs. or more. At smaller unit stresses direct tensile stresses in the concrete will be more or less in evidence and the moments of inertia which express the moments of resistance will be larger for that reason.

A mean value of the moment of inertia of a panel width of the slab may be written in the form

$$I_3 = id^2j\,\Sigma A \text{..............................} (16)$$

in which ΣA is the total cross section of both sets of crossed rods parallel to the sides.

A somewhat larger value of I_3 which may be used as an estimate of its largest possible value, which may be used for purposes of comparison, may be calculated in taking $j\,\Sigma A = 6A_1$ as given in (41) p. 189 of Concrete Steel Construction, edition of 1914, in which A_1 is the cross section of one side belt. Since (41) refers explicitly to the cross section at the cap the values of i, j and d will not be the same when (16) refers to this section as when it refers, as first proposed, to a mean estimated section across the panel any where. This will appear later in the numerical examples.

Let the supporting columns be of the ordinary pattern in their spiral and vertical steel reinforcements. In calculating the moment of inertia of each column two quite different cases must be distinguished from each other, viz. The first and most unusual case is where tension actually occurs under unbalanced load in the concrete in the extreme fiber of the column just below the cap. This may take place in columns supporting deck slabs, because the total compression in such columns is less than in columns having several stories above them, and where greater unbalanced moments are brought to bear upon them because of the absence of any columns above the slab. In this case of actual tension in the extreme fiber at that point of the column where the moment is greatest, the tensile resistance of the concrete must be entirely neglected, so that in calculating the moment of inertia of the column the vertical steel alone must be taken as affording whatever resistance the column as a whole offers to the unbalanced moment, altho since in fact the extreme fiber is not in tension thruout the entire length of the column there may be a somewhat greater moment of inertia operative in part of the column length than that arising from vertical steel alone. The algebraic expression for the moment of inertia of the vertical steel is

$$I_2=\tfrac{1}{2}A_2\, r_2^2 \text{..................................} (17)$$

in which A_3 is the cross sectional area of the vertical steel, and r_3 is the distance of the axis of each vertical rod from the axis of the column. This formula assumes that the moment of inertia of the verticals is the same as tho the same amount of steel were uniformly distributed around the column in the form of a thin cylinder of radius equal to the distance of the centers of the rods from the axis of the column, an assumption that is correct for vertical rods equally spaced around the inner circumference of the spiral.

The spirals do not add to the moment of inertia because they are not under tension except to a very limited extent under ordinary test loading unless it happens that during construction the filling and removal of the column forms has been conducted in such a manner as to develop considerable hydraulic circumferential stresses in the spirals near the column heads. Where that is the case and such stresses are developed in the upper part of the columns just below the column caps the moments of inertia may be somewhat increased by the spiral steel, but the effect had best be regarded as part of that included in the reduction of length already mentioned as negligible.

It thus appears in this first case that the rigidity at O is not only reduced by the absence of any column above the slab but also by a reduction of the rigidity in the column below by reason of the ineffectiveness of the tensile resistance of concrete, a condition of things which, for a given value of M_1, abnormally increases M_3 and v, a result of much importance in the design of deck slabs for bridges, etc., which is further complicated by the fact that these structures are usually entirely made up of outside or exterior panels in which as we have already pointed out every load whatsoever gives rise to unbalanced moments and to no others.

The second case is the one usually occurring in practice where the column is in compression thruout every cross section of its entire length, and the effect of the unbalanced moment is to increase to some extent the compression at one side of the column and decrease it at the other. In this case the compressions in the concrete act in parallel with those in the vertical steel, while the effect of the spirals may be neglected as in the first case. Hence the moment of resistance of the column is the sum total of that of the vertical steel and the concrete added together, the steel being reckoned as it was in the concrete added together, the steel being reckoned as was done in the first case and the resistance of the concrete is that due to the full outside diameter of the column, and not merely that due to the core of the column inside the spiral which alone must be considered in testing columns to destruction.

In this case the value of I_2 to be employed in equations (12), (15) etc., will consist of two parts as follows:

$$I_2 = \tfrac{1}{2} A_s r_s^2 + \tfrac{1}{4} A_c r_c^2 / n \text{ ------------------------ } (18)$$

in which the first part for the steel is the same as (17), and the second part for the concrete is the product of A_c the actual cross section of the concrete acting by $\frac{1}{4} r_c^2$ the square of the radius of gyration of the circular cross section of the concrete having a radius r_c, this product being reduced to a parity with steel by dividing it by $n = E_c/E_s$ as is necessary to be done in order to use it in conjunction with any moment of inertia of steel sections.

8. The Northwestern Glass Company Building. *The distance between centers of the ⅜ inch round slab rods in each of the four way belts is 6.2 inches and the size of the panels is 16 by 17 feet between column centers, and if the entire area be assumed to be uniformly covered with two way reinforcing parallel to the sides as fully equivalent to the actual reinforcement of all except the column heads, that would make 31 rods one way and 33 rods the other, with a total section of 64 crossed rods, or $\Sigma A = 64 \times 0.11 = 7$ sq. in., and with a mean effective slab thickness of 7 in. The steel ratio regarding all the steel as effective is $\dfrac{2 \times 0.11}{7 \times 6.2} = .005$.

Hence $j = 0.89$ and $i = 0.68$, or the moment of inertia due to the slab is $I_3 = id^2 j\, \Sigma A_s = 0.68 \times 7^2 \times 0.89 \times 7 = 207$. This is to be regarded as an outside value of I_3 and will consequently give an outside value of M_3/M_1 in (12).

It is possible, however, also to calculate an outside value of the moment of inertia of the slab from the cross section of the panel at the column head by assuming that $j\, \Sigma A = 6 A_3$ in which $A_3 = 15 \times 0.11 = 1.66$ is the cross section of one belt of fifteen ⅜ in. round rods, $d_3 = 6.5$ in., and $i = 0.5$, so that we have in this case

$$I_3 = 0.5 \times (6.5)^2 \times 6 \times 1.66 = 210,$$

which agrees well with the preceding estimate.

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*See Trans. Am. Soc. C. E., Vol. LXXVII P. 1340, 1914.

This last may be regarded as the largest admissible value of the moment of inertia of the slab, because it assumes all the slab and head steel to act as a single sheet whatever the actual direction of the rods may be as compared with the resisting stresses, and furthermore assumes that the moment of inertia is as large everywhere in the slab as at the edges. The actual moment of inertia of the slab is no doubt somewhat less than that just calculated. Any such smaller value of I_3 would serve to decrease our estimated value of the fraction M_3/M_1 of the unbalanced applied moment resisted by the slab and to increase that of the fraction resisted by the columns.

The moment of inertia of the vertical steel in each of the columns that lie below the floor is that due to six 1⅛ inch round rods equally spaced around the inside of the spiral 27 inches in diameter. Assume the verticals to be situated at a distance $r_s = 13$ inches from the axis of the column. Since each rod has a cross section of approximately 1 sq. inch, the moment of inertia of the verticals is

$$\tfrac{1}{2}A_s r_s^2 = \tfrac{1}{2} \times 6 \times 169 = 507$$

The columns above the floor each have four 1⅛ inch round rods at a distance of 12 inches from the column axis, hence their moment of inertia is

$$\tfrac{1}{2}A_s r_s^2 = \tfrac{1}{2} \times 4 \times 144 = 288.$$

Since the external diameter of the columns below the floor is 28 in., the moment of inertia arising from the concrete is

$$\tfrac{1}{4}A_c r_c^2/n = \tfrac{1}{4}\pi r_c^4/n = \tfrac{1}{4}\pi \times 14^4/15 = 2000$$

Hence for a column below the floor the total moment of inertia is

$$I_2 = 500 + 2000 = 2500.$$

Similarly for a column above the floor having an external diameter of 26 inches the moment of inertia of the concrete is

$$\tfrac{1}{4}\pi \times 13^4/15 = 1500$$

Hence the total moment of inertia of a column above the floor is

$$I_4 = 288 + 1500 = 1788$$

Also $L_2 = 9.75 \times 12 = 117$ in.

$L_3 = 17 \times 12 = 204$ in.

$L_4 = 16 \times 12 = 192$ in.

Substituting these values in (12), we have

$$\frac{M_2}{M_1}=\frac{I_2/L_2}{I_2/L_2+I_3/L_3+I_4/L_4}=\frac{2500/117}{2500/117+210/204+1788/192}=$$

$$\frac{21.4}{21.4+1+9}=0.68$$

and similarly

$$\frac{M_3}{M_1}=.03 \text{ and } \frac{M_4}{M_1}=0.29$$

It is evident, since the outside estimate of the largest possible value of the resisting moment M_3 of the slab is a negligible fraction of the total applied moment M_1 that the loaded panel is practically fixed at the columns by the resisting moments of the columns themselves, and that the ratio of the moment M_3 at O, the margin of a panel, to the moment, M_0 say, at the mid section of the loaded panel is practically the same as the ratio of the end moment to that at mid span, in a continuous beam fixed at the ends. In such a beam the negative moment at the end is $WL/12$ and the positive moment at mid span is half as much. In a panel the applied bending moments at a marginal section and a mid span are less than half these amounts as has been shown previously. Again, with only a single panel loaded as was the case in this test, with one unbalanced load, the unbalanced applied moment M_1 is resisted by two supporting columns instead of one as would be the case were a single bay of panels loaded with an unbalanced load. Consequently the unbalanced moment in each of the two columns is on this account only one half as great as in case of a column supporting a beam of which 68/100ths should be resisted by the column below the floor in the Northwestern Glass Building, as already seen. Hence it appears that the unbalanced resisting moment of a single column in the direction of a side computed according to our theory amounts to

$$\frac{0.68\,WL}{4\times 12}=\frac{WL}{70}$$

Now so far as could be determined, with some uncertainty due to faulty readings in the test of this floor, the unbalanced bending moment resisted by the column under the floor in the direction of the diagonal was $\frac{WL}{58}$* of which the component along a side would be

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*See Trans. Am. Soc. C. E. Vol. LXXVII P. 1718 (1914).

$\frac{WL}{58\sqrt{2}}=\frac{WL}{82}$ a quantity of the same order of magnitude as that just calculated. The agreement is sufficiently close to give probable confirmation of the theory which has already been developed.

Tests that admit of numerical comparison with this theory are as yet too few to supply all the confirmation that is desirable, nevertheless the wide range of variation in the relative values of I_2/L_2 and I_3/L_3 which is available renders a general quantitative confirmation of (12) and (15) attainable, because the various designs range all the way from that of the Northwestern Glass Company Building, where the columns practically fix the corners of the panels horizontally by resisting practically the entire unbalanced moment, to deck floors with small supporting columns of little rigidity all of which are included in the foregoing theory and to which very serious attention should be given in design in order to make provisions for resistances not otherwise required.

9. The Schulze Baking Company Building, at 55th Street and Wabash Avenue, Chicago.* Size of panels 20 by 17½ ft., *i. e.*, $L_1=240$ in., and $L_2=210$ in. Drop head 14 in. thick for an area of 7½ by 7½ ft. about each column. Floor is 9 in. thick elsewhere. Four way reinforcement of 7/16 in. round rods 23 rods on each long side, 20 rods in each diagonal belt and 17 rods on each short side of the panel, or an average 20 rods each way or a mean cross section of $20\times0.15=3$ sq. in. in each belt. One third of the rods in each side belt are carried thru the column heads at the level of the bottom of the slab. All other slab rods are bent up to the top of the slab over the columns. There are 10 or 12 round ⅜ inch rods across each side of the panel in the top of the floor 12 inches between centers and 9 ft. long, which may be regarded as furnishing as much resistance to negative moments as the one third of the rods in the belts which were not bent up over the columns would have furnished had they been bent up over the column heads.

The mean distance between centers of slab rods in the four way belts is 5 inches which, assuming a uniform two way reinforcement would give 48 rods one way and 42 the other way, with a total cross section of 90 crossed rods or $\Sigma A_s=90\times0.15=13.5$ sq. in., with an effective thickness at mid span of 8 in.

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*See Bulletin, No. 84, Eng. Expt. Station, Univ. of Ill., 1916, by Talbot and Slater, p. 61.

Regarding all this steel as effective this would give a steel ratio of $\frac{2\times 0.15}{8\times 5}=.0075$. Hence $j=0.875$ and $i=0.62$, thus that part of the moment of inertia due to this slab steel would be

$$I_3=id_s^2\, j\, \Sigma A_s=0.62\times 8^2\times 0.875\times 13.5=470$$

If instead of the above assumption the moment of inertia of the floor be calculated from the section next to the head in a manner similar to that previously employed we should have

$$j\,\Sigma A_s=5A_3$$

in which we have used the coefficient 5 instead of 6 by reason of the smaller amount of head steel than in regular Mushroom construction.

Then $A_3=20\times 0.15=3$ sq. in. is the cross section of one belt of twenty 7/16 in. rods, $d_3=14-2=12$ in., and $i=0.5$. Hence $I_3=0.5\times 12^2\times 5\times 3=1080$ is the moment of inertia of the slab provided it be assumed to have a uniform thickness of 14 inches across the entire span of each side. But it has that thickness for a distance of only 95 inches out of a mean span of 225 inches. The remaining part of the side not only lacks thickness but any mat of steel on the tension surface, having merely a set of parallel ⅜ inch rounds at about 12 inches between centers.

The total effect of this arrangement is to reduce the effective moment of inertia at the margin in about the same ratio as the effective width of the moment section is reduced at the margin, viz., in the ratio of 225 to 95. Hence the approximate value of the moment of inertia does not exceed $I_3=460$, a value which differs little from that at mid span previously obtained.

The length of the columns is 18 ft. 8 in., from floor to floor. The outside diameter of the columns below the second floor on which the test was made is 28 in., with 24 inch spirals made of ½ inch rounds, enclosing twelve 1⅛ in. square verticals. The columns above this floor are 26 in., outside diameter with 22 in. spirals of ½ in. rounds and twelve 1 in. square verticals.

The moment of inertia of the vertical steel in the columns that lie below the slab is that due to twelve 1⅛ in. square verticals equally spaced around the inside of a 24 inch spiral. The moment of inertia of these verticals is $\frac{1}{2}A_s r_s^2$ in which $A_s=12\times(\frac{9}{8})^2=15.2$ sq. in., and $r_s=11.5$, hence

$$\tfrac{1}{2}A_s r_s^2=1005$$

while for the column lying above the slab $A_s = 12$ sq. in., and $r_s = 11.5$. Hence

$$\tfrac{1}{2}A_s r_s^2 = 660$$

That part of the moment of inertia in the column below the slab arising from the concrete is

$$\tfrac{1}{4}A_c r_c^2/n = \tfrac{1}{4}\pi r_c^4/n = \frac{\pi}{4}\times 14^4/15 = 2000$$

if diameter of column = 28 inches.

Similarly if the column above is 26 in. outside diameter we have

$$\tfrac{1}{4}A_c r_c^2/n = \frac{\pi}{4}\times 13^4/15 = 1495.$$

Hence the total moment of inertia of the column lying below the floor is $I_2 = 1000 + 2000 = 3000$ and that of the column above the floor is $I_4 = 660 + 1490 = 2150$. $L_2 = L_4 = 18$ ft. 8 in. = 224 in.

The mean span $L_3 = \tfrac{1}{2}(20 + 17\tfrac{1}{2})12 = 225.$ in.

Assume $L_2 = L_3 = L_4$ and by (12)

$$\frac{M_2}{M_1} = \frac{I_2}{I_2 + I_3 + I_4} = \frac{3000}{3000 + 450 + 2000} = 0.55$$

$$\frac{M_3}{M_1} = \frac{1}{12} = 0.08 \quad \frac{M_4}{M_1} = 0.37$$

In case the columns are integral with the slab the resisting moment M_3 of the slab exceeds one twelfth of M_1 the applied unbalanced moment. This value of M_3 is an outside estimate larger than can actually occur for a slab rigidily joined to the columns because of the very liberal allowance for the moment of inertia of the slab steel in I_2. The only point of real doubt lies in the connections between slab and columns which apparently are less rigid in the Schulze floor than in the floor of the Northwestern Glass Company Building, because there are no such direct steel connections in the shape of elbow rods, etc., as in the latter building. It does not seem possible that the drop head can furnish connections of the same rigidity as are found in steel framework of the Northwestern Glass floor. If that be the case M_2 and M_4 would be diminished thereby and M_3 would be increased by this lack of almost perfectly rigid connections between floor and columns.

The observed deformations of slab steel in the test of the Schulze floor are too few and irregular to furnish a conclusive proof but the results completely accord with the above theory, as will appear from the following computation.

The loaded area covered a square of four panels and col. 9 was at a corner of this area. Gage lines 274 and 275 were in parallel slab rods of a diagonal belt across the edge of a column cap at col. 9 just outside the loaded area. Gage lines 276, 277 and 278 were in parallel slab rods of the same diagonal across the edge of the same cap at col. 9, but just inside the loaded area.

In case the elongations at these gage lines exhibited no irregularities we should expect to find that the ratio of the elongations at 274 and 275 would have the same ratio to those at 276 and 278 as M_3 to M_1. But it is to be noticed that while the elongations at 276, 277 and 278 under the load increased pretty regularly with the increase of the load and a much smaller regular increase occurred at 274 outside the load, no such thing occurred at 275, where as light final compression was indicated rather than a tension, so that it is impossible to say for certain which furnishes the more reliable indication, 274 or 275. The probability, however, seems to be that the shrinkage stresses of the concrete in the cap and drop produced more or less initial compression of the steel and that the small elongation observed at 274 was the relief of that compression.

Besides the observations just mentioned, observations of elongations were made on the middle rod of each of the rectangular belts that cross the middle of the loaded area lengthwise and crosswise respectively. Observations were made at the middle of each side of the loaded area under the load where the middle rod crosses the edges of the cap at gage lines 268, 272, 281 and 283. While these elongations all increase with the load, they all increase more rapidly at first than afterwards, thus showing unmistakably the effect of initial shrinkage. There is only a single observation, viz., at 267, upon one of these rods outside the loaded area where it crosses the column cap and it shows no increase of elongation during the addition of the last half of the load.

The indications are therefore that after the initial shrinkage has been overcome the portion of the unbalanced load which is resisted by the slab itself is practically negligible, a result which would be in accordance with the theory which has been proposed as applied to this floor.

10. The Winston-Salem Factory Buildings. Results of tests made on floors of factory buildings Nos. 63 and 64 reported by Mr. Earle Mauldin, Engineer, furnish a decisive check on the accuracy of the foregoing analysis because of the wide difference in degree of restraint afforded by the columns in the two respective cases. The floor slabs in these buildings were identical so far as the thickness of the slab and the reinforcement are concerned, but the size, length and reinforcement of the columns differ widely, one being a deck slab supported on long slender columns, the other a slab supported by continuous large rigid columns thoroughly restraining the edges of the panel. The arrangement of the reinforcement was that of the Spiral Mushroom design, panels 22 by 22′, slab thickness 9¾″ in the rough, with 36 round 7/16″ rods parallel to each side, besides which there were belts of six oblique rods of the same size, 10″ from center to center crossing the panel sides at mid span each way and lying at the bottom of the slab. Each column head has a flat spiral with six turns 9′0″ in outside diameter of ½″ round supported by four 1″ radial rods.

In order to calculate I_3, the moment of inertia of a panel section at mid span, take first the total cross section of both belts of 12 rods that cross each other in contact in the central area of the panel as if they constituted a single sheet of metal and add to this the belt of 24 rods cut by this mid section and the two oblique belts of six rods each, making 60 rods all told. Consequently the total mid section amounts to $\Sigma A = 60 \times 0.15 = 9$ sq. in. The mean distance between centers of slab rods is 6.3 in., and if the effective thickness of the slab is taken to be $d = 8.5$ in., the steel ratio is

$$\frac{2 \times 0.15}{8.5 \times 6.3} = .0056$$

from which $j = 0.89$ and $i = 0.67$ and we have

$$I_3 = id^2j\,\Sigma A = 0.67 \times 8.5^2 \times 0.89 \times 9 = 388$$

If the moment of inertia of the panel be calculated instead by taking a section at the margin we find by applying the formula $I_3 = id^2j\,\Sigma A$ separately to the steel in the column head and to the steel intermediate between the column heads that we may assume so far as concerns the two way steel and spiral, etc., in the column head, that in this case

$$j\,\Sigma A = 3A_1 = 3 \times 24 \times 0.15 = 10.8 \text{ sq. in.}$$

and

$$id^2j\,\Sigma A = 0.6 \times 7^2 \times 10.8 = 317.5$$

and for the belt of 12 intermediate rods that crosses the side between the column heads

$$id^2j\,\Sigma A = 0.65 \times 8^2 \times 0.9 \times 12 \times 0.15 = 67.4$$

Hence $\quad I_3 = 317.5 + 67.4 = 385$

a result in close agreement with that previously obtained for the value of I_3 at the mid section.

Building No. 64 is five stories high and the columns both above and below the third floor on which the test was made are 14 feet long and 28 inches outside diameter with spirals 25 inches in diameter. There are six 1⅛ inch round vertical rods inside the spiral below the third floor and four 1⅛ inch round verticals inside the spiral above the floor. The moment of inertia of the vertical steel in a column under the third floor assuming the vertical steel to be situated at a distance $r_s = 12$ inches from the axis of the column will be

$$\tfrac{1}{2}A_s r_s^2 = \tfrac{1}{2} \times 6 \times 144 = 432$$

and that of the concrete is

$$\frac{\pi}{4} \times 14^4/15 = 2015.$$

Hence $\quad I_2 = 432 \times 2015 = 2447$

Similarly the moment of inertia of the vertical steel in the column above the floor is

$$\tfrac{1}{2}A_s r_s^2 = \tfrac{1}{2} \times 4 \times 144 = 288$$

Hence $\quad I_4 = 288 + 2015 = 2303$

Consequently

$$\frac{M_3}{M_1} = \frac{285/22}{385/22 + (2447 + 2303)/14} = \frac{17.5}{17.5 + 340} = 0.05$$

from which it appears that only 5 percent of the unbalanced applied moment is carried by the slab and 95 percent by the columns, *i. e.*, the slab is almost completely fixed by the rigidity of the columns.

Building No. 63 is a shed of but one story and the floor is a deck slab with columns 18 ft. long under the panel carrying the unbalanced load, and having an outside diameter of 18 in., reinforced with four one inch round verticals inside a 15 inch spiral.

Since any heavy unbalanced load will produce tension on the extreme fiber of this column the moment of inertia of the column will include only the vertical steel of the column, viz.,

$$I_2 = \tfrac{1}{2}Ar^2 = \tfrac{1}{2} \times 4 \times 0.7854 \times 7^2 = 77$$

Consequently

$$\frac{M_2}{M_1}=\frac{77/18}{77/18+385/22}=0.2$$

and

$$\frac{M_3}{M_1}=0.8$$

in which case the slab is called on to resist four fifths of the applied unbalanced moment and the column only one fifth.

These floors were each subjected to test loads of 600 lbs. per square foot upon a single interior panel with the result that the central panel deflection of No. 63 was 1″ and 15/32″ = 1.47 in., while the central deflection of the loaded panel in No. 64 was 0.69 in. The deflection of the deck slab in No. 63 was 2.16 times that of the standard floor in No. 64, a result that might have been expected from our investigation of the relative deflections of beams, as found in Section 3 of this Chapter, for it is evident that altho the actual moments at mid section and margin of a panel is not the same as at the middle and end of a beam due to a given load, nevertheless the ratio of the moment at mid span to the moment at the supports will be the same in the two structures if the restraints are alike, and the same relative changes in central deflections will occur in the two structures for like relative changes in restraints at the supports. In other words if in the beam discussed in Section 3, a change from fixity at *B* and *O* as in Case 1 to freedom at *B* and *O* as in cases 2 and 3, more than doubles the deflection, the same change of fixity in a slab would also more than double the deflection. That in fact is the difference between the floors in No. 64 and No. 63. In No. 64 the panel is almost completely fixed at the corners of the loaded panel by the rigidity of the column, but in No. 63 the rigidity of the columns is small and the panel is only slightly restrained by the columns at the corners.

As already stated the applied unbalanced moment which in No. 64 is almost entirely resisted by the columns is so reduced in No. 63 that only one fifth is resisted by the column, leaving four fifths to be resisted by the slab itself. The effects of this transfer of the principal resistance from columns to the slab itself in case of large concentration of unbalanced load in a single panel, or in several panels, are many and important. They depend to some extent upon the location and arrangement of the reinforcement.

The first notable effect of a large uniform load placed upon a single panel is to increase the central deflection to a greater extent than would occur if an equal load were placed at the same time upon the surrounding panels. This larger deflection is accompanied by a tipping downward of the panel over the columns at the corners toward the center of the loaded panel and a movement of lines of inflection away from the panel center, *i. e.*, an enlargement of the suspended span in this panel at the expense of the surrounding cantilever portions of the panel. An increase of the length and width of the suspended span by 15 percent would be sufficient of itself to double the central deflection and add 50 percent to the bending moments at the center, as is evident when we consider that deflections in a square panel depend on the factor $WL^3=wL^5$ in which w is the load per unit of area, and that $(1.15)^5=2$, while moments depend on the factors $WL=wL^3$, and $(1.15)^3=1.5$.

It is important consequently to prevent any such movement of the lines of inflection or at least to limit their movement as much as possible. This may be accomplished by having all the reinforcing rods slope from top to bottom of slab at a predetermined line of inflection where the moment of inertia is greatly reduced by this arrangement, so much so as to produce nearly the same kind of effect as a joint does at the end of a cantilever in a cantilever bridge. When that has been done, it is necessary, in order to prevent any large increase of the bending moment at the panel center beyond that due to an equal load on all panels, that the unbalanced moment at the edge of the panel should be provided for either by larger and stiffer columns or by so increasing the reinforcement which connects the successive column heads as to furnish so much resistance to twisting that near by columns assist in resisting the bending moment at the edge as included in equation (15).

It is to be noticed that unbalanced loads tend to greatly increase the bending moments at mid span, beyond those due to balanced loads of equal amount while they tend to decrease the bending moments at the edges of the panels below those due to balanced loads of equal amount, and the preceding recommendations are for the purpose of obviating so far as may be such excessive moments and the deflections that accompany them.

Excessive central deflections are accompanied by excessive tipping and tensile stresses on the extreme fiber of columns which should be avoided by suitable provisions to reduce the resisting moments of the columns by limiting the movement of lines of inflection which may be done by increasing the rigidity of the columns themselves, and by providing twisting rigidity in the slab along the edges of the panels connecting the columns, but not by concrete beams connecting the column heads because that is an ineffective device for that purpose, as has been recently observed in a couple of instances, where spiral cracks developed under test loads. One of these is figured in the Engineering News.*

The importance of avoiding tension in the extreme fiber of columns is evident when we consider that the rigidity and moment of resistance of a column is several times as great with no tension in the concrete as when tension is allowed to occur. When this untoward condition does occur a marked and dangerous increase of slab deflection and column bending usually takes place which needs careful attention.

The initial bending moments may then be computed either by assuming a suitable increase in the span commensurate with the enlargement of the suspended span arising from the movement of the lines of inflection as already pointed out, or by the theorem of three moments as already employed for that purpose.

11. The St. Paul Bread Company Building.* This is a Turner Mushroom building with a rough slab 6 in. thick. The panels are 16 by 15 feet between column centers reinforced by ⅜ inch round rods in each belt. The columns under the floor are 11 ft. long and 20 inches outside diameter enclosing a 17 inch spiral within which are four 1⅛ inch round verticals, while the columns above the floor are 12 ft. long and 18 inches outside diameter enclosing a 15 inch spiral within which are four 1⅛ inch round verticals.

Compute as in the case of the Northwestern Glass Company building previously. The mean distance between centers of the ⅜ inch slab rods in each of the four way belts is 9 inches. If the entire area be assumed to be uniformly covered with two way reinforcement 9 in. between centers parallel to the sides as fully equivalent to the actual reinforcement of all except the column heads, that would make 21 rods one way and 20 rods the other way with a total section of 41 crossed rods, or

$$\Sigma A = 41 \times 0.11 = 4.5 \text{ sq. inches.}$$

Assume a mean effective slab thickness of 5 inches then the steel ratio, if all the steel be regarded as effective is

$$\frac{2 \times 0.11}{5 \times 9} = .005$$

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*Deflection and Wall Girder Tests in Flat Slabs, etc., by C. E. Luck, Engineering News, Feb. 1, 1917, p. 176.

*See Trans. Am. Soc. C. E., Vol. LXXVII P. 1376 (1914).

Hence $j=0.89$ and $i=0.68$, or the moment of inertia due to the slab is

$$I_3 = id_s^2 j\, \Sigma A = 0.68 \times 5^2 \times 0.89 \times 4.5 = 68$$

If the moment of inertia of the slab be calculated at the margin we have as previously in the Northwestern Glass Company building.

$$I_3 = 0.5 \times (4.5)^2 \times 6 \times 1.1 = 67$$

The moment of inertia of the verticals below the floor is

$$\tfrac{1}{2} A_s r_s^2 = \tfrac{1}{2} \times 4 \times 64 = 128$$

The moment of inertia of the concrete in the columns below the floor is

$$\tfrac{1}{4} A_c r_c^2 / n = \tfrac{1}{4} \pi r_c^4 / n = .7854 \times 10^4 / 15 = 524$$

The moment of inertia of the verticals above the floor is

$$\tfrac{1}{2} \times 4 \times 49 = 98$$

The moment of inertia of the concrete in the columns above the floor is

$$.7854 \times 9^4 / 15 = 344.$$

Hence $I_2 = 128 + 524 = 652$

and $I_4 = 98 + 344 = 442$

$$\frac{M_3}{M_1} = \frac{67/15.5}{67/15.5 + 652 + 442/12} = \frac{4.3}{4.3 + 59.3 + 38.6} = .04$$

Hence $\dfrac{M_2 + M_4}{M_1} = 0.96$

This shows that the applied bending moment is almost all of it resisted by the columns and very little of it by the slab, *i. e.*, almost complete fixity occurs at the corners of the panels. This may be contrasted with the floor of International Hall of somewhere nearly equal span and thickness but far less rigidity of column resistance.

12. International Hall, Union Stock Yards and Transit Company, Chicago. This is a deck floor slab consisting of three bays of nine panels each, built in at the edges into old brick walls. The panels are 18 by 18 feet and the rough slab was 7 inches thick of four way Turner Mushroom design with thirteen $\frac{3}{8}$ inch round rods in each belt 6 inches between centers.

The columns are 16 inches external diameter enclosing a spiral 13 inches in diameter with eight 1 inch round verticals. Columns 12 ft. 9 in. long to a basement floor resting on a fill four feet thick, Considering the effective reinforcement at mid span to be equivalent to 36 round $\frac{3}{8}$ inch rods parallel to each side the total cross section of an equivalent sheet of steel is $\Sigma A = 72\times 0.11 = 7.9$ sq. inches. Assume a mean effective slab thickness of 6 inches, then the steel ratio, if all the steel be regarded as acting is

$$\frac{2\times 0.11}{7.9\times 6} = .0046$$

Hence $j = 0.895$ and $i = 0.65$, or the moment of inertia due to the slab

is $\quad I_s = i d_s^2 j\, \Sigma A = 0.65\times 6^2\times 0.895\times 7.9 = 166$

Again, $\quad I_2 = \frac{1}{2} A_s r_s^2 = \frac{1}{2}\times 8\times 0.7853\times 6^2 = 113.$

The concrete in the column being subject to tension does not add to its moment of inertia.

We have by (12)

$$\frac{M_3}{M_1} = \frac{I_3/L_3}{I_3/L_3 + I_2/L_2} = \frac{166/18}{166/18 + 113/12.75} = \frac{9.2}{9.2 + 8.86} = 0.51$$

$$\frac{M_2}{M_1} = 0.49$$

in which the resisting moment of slab and column are approximately equal.

In the final test of this floor two adjacent panels of the middle bay at the center of the floor were loaded with a heavy unbalanced load. This imposed very severe bending moments upon the two columns at the middle of each edge of the loaded area, but owing to poor foundations there was a considerable amount of permanent subsidence and tipping of these columns. These complications render it impossible to precisely compare the deflections in this test with our proposed theory, but in conjunction with the theory they account for the large deflections that occured in this test.*

13. Unbalanced Test Loads and Modification of Slab Steel Stresses by Column Bending in Column Supported Slabs. Test loads applied to floors are usually double the rated working load or sometimes twice the working load plus the amount of the dead load all applied over one or more panels. The load is commonly placed on four panels about a column and then doubled up over two of them, these two being sometimes adjacent to each other side by side and sometimes adjacent to each other diagonally. These various cases will accordingly be compared as to the amount of the unbalanced moments they bring to bear upon the columns at the edge of the load and at mid span.

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*See Jour. Franklin Inst., Dec. 1916, page 761.

I. In a single panel test the number of supporting columns is four. One quarter of the panel load is an unbalanced load with respect to each of the four supporting columns and the tendency of each of the columns is to bend toward the center of gravity of the load.

II. When four adjacent panels are loaded, there are nine columns all told which support the test load. There is no unbalanced load, whatever, about the central column. Each of the four columns at the middle of each side has approximately half a panel of unbalanced load and each corner column has an unbalanced load of one quarter of a panel.

III. When the load of Case II over four panels is doubled up on diagonally adjacent panels there are seven supporting columns and no unbalanced load is brought upon the center column. Approximately one quarter of a panel of unbalanced load is brought on each of the other six supporting columns.

IV. When the test load covers two adjacent panels side by side there are six supporting columns and the load is divided approximately in such wise that the unbalanced load on the two central columns is a half panel load, and on each of the other four a quarter of a panel load only. This division being approximately exact for fairly stiff columns.

Any unbalanced load causes an amount of bending in the columns that is inversely proportional to the stiffness of the columns which last is proportional to the moment of inertia of the columns. Thus in a column heavily loaded by other than the unbalanced panel load the concrete may resist through its entire section by a mere shift (due to the bending) of the intensity of the uniform compression. Where, however, the load, other than the unbalanced load is so small that the bending brings tensions upon part of the steel in the cross section, the moment of inertia of the column is reduced to that of the steel on the tension side and the stiffness of the column is reduced in proportion to the reduction of the moment of inertia. Any such reduction of the moment of inertia increases the angle v.

It is perfectly possible to apply the methods of analysis that have been employed in this chapter to each and all of these cases to derive exact solutions for them. The following method, however, will bring out points of importance with sufficient accuracy to permit of the comparison of the effects of the several cases of test loading that have been enumerated.

In case of an unbalanced moment at the edge of a load such as arises in cases I to IV where the moments of inertia of the supporting columns are so large in proportion to that of the panel itself as to afford nearly perfect restraint at the edge of the load, it is possible to make a close estimate of the relative magnitudes of the unbalanced moments produced at mid span in the several cases previously described. The magnitudes of these unbalanced moments at mid span are independent of the magnitude of the total moments at mid span, and refer only to the corrections that are to be made to the values they would have were all panels uniformly loaded, by reason of the load being confined to one, two or four panels as described in Cases I to IV instead of being uniformly distributed on all panels. At the edge of a load next to a column the difference between the moment that actually exists and that which would exist were all panels equally loaded is the unbalanced moment and it changes the slope of the slab and column by some angle v which is directly proportional to the magnitude of the unbalanced moment, v being the direct effect and measure of its action.

A change of slope at a column at one edge of a test load which is due to an unbalanced moment at the point will produce a change of slope throut the span which will gradually diminish in amount to the opposite column if there is none at that column. The change of slope will diminish uniformly to the column opposite and the change of slope at mid span will be half that at the edge, if we assume that the moment of inertia is practically constant thruout the span. Hence the unbalanced moment at mid span will be half that introduced at the edge.

In case II where four adjacent panels are loaded the unbalanced load will have approximately twice as great an effect to bend each of the four columns at the middles of the sides toward the center column as it will have to bend the corner columns in the same directions. But each corner column being subjected simultaneously to two flexures at right angles to each other, will have a resultant bending toward the center column such that if v designate the bending of a column at the middle of a side and $\frac{1}{2}v$ that component of the bending of a corner column that is parallel to it, then $v/\sqrt{2}$ is the resultant bending toward the center column of a corner column.

In Case III when half the loading of Case II has been removed by taking the load from two diagonal panels of the four loaded in Case II it is evident that each side column in Case II owes one half of its bending v to one pair of diagonal panel loads and one half to the other pair. It is therefore clear that in Case III each of the seven supporting columns except the center one is bent approximately in the direction of each of the sides by an amount $= \frac{1}{2}v$ and to a resultant amount $= v/\sqrt{2}$ toward the center of the loaded panel it supports.

In Case I the same may be taken as true respecting each of the four supporting columns but in this case the opposite slopes at mid span cancel each other so that no unbalanced moment exists in this case at mid span. Nevertheless, the resultant bending moment at mid span has been increased by half the sum of the unbalanced bending moments at the edges of the panel.

In Case IV of two adjacent panels on six columns the bending at the four corner columns is approximately $\frac{1}{2}v$ in each direction, but at the two middle columns it is v towards each other from which results the unbalanced moments at mid span and edges of load are seen in all these cases to be measured either by v or $\frac{1}{2}v$ in the direction of the sides.

14. Summary. The subdivision and distribution of the moments applied to any structure are controlled by the relative rigidities of its parts: and the theory of beam structures is at present so fully developed as to permit the thoro investigation and discussion of almost any arrangement of parts that may be proposed, whatever may be its connections and continuities. But the magnitudes of the resisting moments and stresses in flat slab structures due to a given loading is a subject upon which there is not as yet so satisfactory an agreement among engineers as respecting beam structures. Nevertheless, since the distribution of moments and stresses in flat slabs depends upon the relative rigidities just as it does in beam structures the relative magnitudes of the resisting moments in the several parts of flat slab structures must be the same as tho they were beam structures altho the actual magnitudes of the moments are not the same in flat slabs as in corresponding beam structures.

Advantage has been taken of this fact in this Chapter to investigate the relative magnitudes of the bending moments due to unbalanced loads, in order to calculate what proportion of the total bending moment due to unbalanced loading is resisted by columns integral with a continuous flat plate floor, and what proportion by the floor itself. In doing this the proportionate parts in case of a corresponding beam and column structure are first investigated as dependent upon the relative rigidities. The corresponding ratios for flat plate floors and columns are then treated by investigating the moments of inertia which express the rigidities of floor plates and columns.

The formulas so derived have then been compared with experimental results in case of several floor slabs that have been submitted to extensometer tests. The floors tested include the Northwestern Glass Co. Building, Minneapolis, the Schulze Baking Co. Building, Chicago, the Factory Buildings, Nos. 63 and 64 at Winston-Salem, N. C., the St. Paul Bread Company Building and International Hall Chicago. The results in all those cases are in such agreement with the proposed theory as to give confirmation to the practical correctness of the theory developed.

Note: In estimating the numerical values of the moments of inertia I_3 of the slab in (16) and elsewhere in the Chapter, it has not been thought necessary to follow the methods employed in the more detailed calculations for moments of inertia in slabs in Chapter X, since the approximate methods here used are regarded as a sufficient basis for a satisfactory comparison of the relative matitudes of the moment of inertia of the slab and the columns. In case greater precision may be thought desirable more exact values of I_3 may be calculated according to Chapter X.

CHAPTER XIII

EXPERIMENTAL VERIFICATION OF THE THEORY OF FLEXURE OF CONCRETE MEMBERS.

1. Analysis. Any acceptible theoretical analysis of a reinforced concrete structure must agree with and be substantiated by experimental results.

Since mathematical analysis of this structure is based upon elastic theory a most convincing proof of its correctness is by the agreement of the elastic deflection of the member under load with the deflection computed by the elastic theory.

Again, the measured deformations properly interpreted should accord with the computed stresses. The external work of the loads may also be computed from measured deflections and the storage of this energy as internal work of deformation should be in accord with that indicated by the theory derived to account for the predominant mode of operation.

In Chapter III it was shown that mathematical theory treats of the predominant action neglecting subsidiary actions. In the verification of such mathematical analysis proper weight must therefore be given to the degree in which the action treated is predominant and the proportional part of the work performed by each subsidiary action in the total work of the applied loads.

In the preceding chapters, the general principles of flexure in transverse bending have been developed. In pure flexure the sum of the horizontal tensions equals the sum of the horizontal compressions but such a distribution of stress assumes no horizontal reactions at the support. It assumes that the reactions of the beam are vertical. The presence of restraining forces which cause horizontal reactions alters somewhat the relative distribution of stresses.

2. Arch Action in Simple Beams. In Chapter II Section 10, arch action in short span slabs was briefly treated. The beams considered were assumed to be reinforced in the bottom throughout and the restraint at the support was that of a horizontal thrust due to a rigid backing that under bending opposes any increase in the distance from end to end of the lower fiber of the beam. Such restraint produces horizontal compressions at the support and a resultant reaction that is inclined instead of vertical. In such a case two kinds of action are present, arch action brought about by the horizontal restraint and beam action of flexurel resistance.

When the depth is small in proportion to the span the arch becomes so flat that the deflections by arch action would be much larger than by beam action. Since the relative rigidity is small in a thin slab under arch action, the principle of rigidities requires that the proportion of load carried thereby shall be correspondingly small. Hence the most of the load would be carried in this case by beam action.

In the article before referred to, it was stated that when the span of a beam is ten times its depth, and the required restraint is present the mid span applied moment which is resisted by the steel in a simply supported beam, viz. $WL/8$ is one half of it carried by arch action and half by beam action, and the actual resisting moment for the steel becomes approximately $WL/16$. This represents closely the condition deduced by experiment where the amount of reinforcement is from five to six tenths of one percent. Where the percentage of the reinforcement is higher than this beam action would be greater than the arch action. For balanced reinforcement, arch action would not equal beam action until the depth of the beam becomes approximately one eighth of the clear span length. For depths greater than this arch action would be increasingly greater than beam action, but the increase would be at a much less rapid rate than for smaller depths. Where the ratio of depth to span is as small as one twenty-fifth, arch action becomes almost negligible and may be entirely neglected.

It thus appears that the percentage of steel enters as a factor in determining the relative flexural rigidity while the relative rigidity of arch action caused by end restraint depends directly on the concrete and increases about as the square of the ratio of the depth relative to span length.

Since the relative rigidity of arch action depends directly on the concrete it increases as the modulus E_c of the concrete increases. This is equivalent to stating that in order for the rigidity of arch action and beam action to maintain the same relation to each other in different beams with the same restraint and same ratio of depth to span but made of different grades of concrete, the more rigid the concrete is the higher must be the percentage of steel to maintain the same relative amount of beam action and arch action in the beams compared.

The general characteristics governing the distribution of stress here discussed are well known and generally recognized although so far as known they have never before been discussed in works on this subject. The effect of end restraint of rigid supports upon the magnitude of deflection will now be considered.

In simple beam action the horizontal stresses approach zero at the support. In arch action the greatest compression occurs at the support. Hence these two actions when superposed on each other bring maximum stresses to bear on considerable material which would be practically unstrained in simple flexure. It produces a state of stress in which the horizontal compressions no longer balance the horizontal tensions as they do in simple beam flexure. The result is that the relative stiffness of the beam where arch action equals beam action is nearly twice as great as it would be were the beam operating by simple flexure. Where the ratio of depth to span decreases, the arch action decreases and the deflection increases and finally becomes nearly the same as that of the simply supported beam where the depth is as small as $L/25$.

3. Arch Action in Continuous Beams. In the continuous beam as in the simple beam, arch action depends directly on the relative rigidity of the concrete as compared with the relative rigidity of the concrete and of the metal coacting in flexure. In the continuous beam the different mode of coaction of the metal and concrete from that of the simple beam renders its relative rigidity five times as great as that of the simple beam and accordingly instead of a ratio of $\frac{1}{2}$ arch action and $\frac{1}{2}$ beam action as in the simple beam where the ratio of depth to span is $1/8$, we would have approximately $5/6$ as the beam action and $1/6$ as the portion of the arch action and this for a value of $E_c = 2.5 \times 10^6$ and a ratio of reinforcement of 1.25 percent.

Where the reinforcement is less than 1.25 percent, the proportion of arch action would be increased. As in the case of the simple beam with the reinforcement in the bottom throughout the percentage of arch action decreases approximately in proportion to the square of the ratio of the depth of the beam to the span length.

4. Arch Action in Column Supported Flat Slabs. In a flat slab of many panels the horizontal distances between interior column and panel centers are to be regarded as invariably fixed by its continuity, hence any given vertical deflections of panel centers will develop horizontal thrusts along these invariable lines whose magnitudes will be dependent upon the relative thickness of the slab and upon the modulus E_c of the concrete when other things are equal. Each of these thrusts at any vertical cross section constitutes one part of the total horizontal compression in the concrete that holds in equilibrium the total moment at that section which arises from the loading. The other part of the compression is equal and opposite to the steel tension across the section. These last together constitute the moment of flexure or bending action existing at that section.

It thus appears that the arch action does not alter the maximum compressions in the concrete, but the horizontal thrust of this arch action does reduce by just so much the tensile stresses which would otherwise be exerted by the steel both at mid span at the bottom and also over the supports at the top of the floor.

In a continuous slab the relation between the rigidity of the concrete in producing arch action and the moment of inertia of the slab in resisting flexure is analogous to the arch action and flexure in a continuous beam.

About the diagonal center of the panel the arch action is of the nature of dome action with radial and circumferential stresses cooperating with each other and inverted dome action about the column in the circumferential cantilever area. Accordingly, we may consider that circumferential action applies alike both to the arch action and to the combined flexural resistance of the concrete and metal, and we may therefore approximate to the separation of these two actions by consideration of continuity only in the same manner as that followed for the continuous beam. The thickness of the slab should, however, be taken as that of the slab plus the thickness of the drop panel, where a drop head is employed, and the span for comparative purposes should be the neat clear span between the outer edge of the column capitals in square panels.

In general the reinforcement over the capital approaches that of balanced reinforcement for a short distance in the vicinity of the cap but at mid span because of the small ratio of the thickness of the slab to the span length the percentage of steel is necessarily less than half that permissible in the deeper continuous beam in order that the curvature may not become too sharp when the steel is stressed to its safe working stress. We may therfore, conclude from the steel ratios, that the arch action is approximately one sixth the applied moment where the thickness of slab is one eleventh of the effective span and that the ratio of arch action to the applied moment decreases or increases in square panels with the square of the ratio of the depth to effective span.

In rectangular panels a mean dimension should be taken as the effective span, which is best arrived at by dividing the sum of the squares of the long and short sides by the sum of these two sides and then deducting the diameter of the capital for the average effective span to which this proportion is to be applied.

In case where the relative span is $N=L/d=11$, when the percentage of reinforcement is 1.25 and $E_c=2{,}500{,}000$, or $n=12$, it has been stated that arch action is $1/6$ of the entire applied moment or 20 percent of the beam action. Now the value of E_c is not constant but decreases with increase of intensity of the stress in the concrete and vice versa, a fact which makes the amount of arch action and beam action both depend upon the value of E_c. It is proposed here to show how to calculate from the foregoing the relative amount of the arch action for any value of E_c whatever. For example to calculate the arch action when the intensity of stress in the concrete is so small that $E_c=3{,}750{,}000$ or $n=8$. The rigidity of the concrete is then 1.5 times as great as when $E_c=2{,}500{,}000$, and the rigidity of arch action is concequently larger in the same ratio. But the rigidity of beam action which is in proportion to the moment of inertia of the section, $I=Aijd^2$, is also greater at the same time because of the proposed alterations in the value of E_c, or of n from 12 to 8, alters the values of both $i=1\text{-}k$ and j, as shown on page 156. It is found that the value of I or of ij is 13⅓ percent larger when $n=8$ than when $n=12$, and the beam rigidity is consequently 1.133 times as great as when $n=12$, *i. e.* the ratio of the rigidities which is $1/5=.20$ when $n=12$, becomes $\dfrac{1.5}{5\times1.133}=.265$ when $n=8$, and has thus been increased by one third, while the arch action instead of being $\frac{1}{5+1}=1/6$ of the applied moment now becomes $\frac{1.5}{5\times 1.13+1.5}=1/5+$.

Thus some part of the increase in steel stress with the increase in load is accounted for by the decrease in arch action accompanying the decrease of the modulus E_c of the concrete. This increase in steel stress caused by decrease in arch action under higher loads is a phenomenon often erroneously ascribed to an elimination of the tensile resistance of the concrete instead of to a decrease in the compressive resistance of the concrete.

Arch action also decreases as the deflection increases since the rise is thus reduced in the ratio of $\frac{md}{md-\triangle}$ and the stiffness in the ratio of $\left(\frac{md}{md-\triangle}\right)^2$ in which m is a fractional multiplier representing the part of the depth which is effective as the rise for arch action and $\triangle$ the deflection.

5. Reduction of Measured Fiber Deformations to Axial Deformation in Reinforcing Rods. Deformations computed according to theory of flexure represent the effect of axial stress. Deformations measured for convenience on the outer fiber of the bar are not axial stresses. Now the steel stress under theory varies as the distance from the fiber to the neutral plane. Consequently, if i is the distance from the axis of the bar to the neutral plane, $i+\frac{1}{2}d$ is the distance to the outer fiber of the bar having a diameter d. The measured stress is greater than the axial stress in the ratio of $\frac{i+\frac{1}{2}d.}{i}$

In case the neutral plane is 4″ from the center of a 1/2″ inch rod, this correction to the measured stress would be a reduction of 6¼ percent. In case it were a 5/8″ rod, the correction would be a reduction of 7.9 percent. In case of the Printers Union Building where there were 5/8″ rods and four layers in an 8″ slab, and the distance to the neutral plane 2½″ or less from the lower layer of the steel, the reduction would amount to 12½ percent for elastic coaction of the metal and concrete and increase to 17 or 18% as the modulus of elasticity of the concrete decreased.

This correction has been commonly disregarded but in any such abnormal design as that of the Printers Union Building to be discussed later, it cannot be disregarded in a rational interpretation of test results.

Sharp Bending. Where there is any yielding of the concrete in compression, sharp bending occurs across the section. This is accompanied by a shifting of the neutral axis toward the steel, a reduction in the lever arm of the steel stress and an increase in the steel stress in proportion to the reduction of its lever arm.

Any reduction in the modulus of elasticity of the concrete produces the relative increases above noted in the fiber steel stress.

At a section of greatest shearing strain a material increase in fiber stress may also come about in another manner. The modulus of combined shearing rigidity is increased by the embedment of steel. The rigidity of the steel in shear is about 10 times as great as that of the concrete in which it is embedded. The steel bar embedded in the concrete tends by its bearing on the concrete to transfer the shearing resistance away from the plane of section where sliding might otherwise occur. In so doing bending stress is brought upon the bar as the shear is transferred by flexure to the centers of bearing in the concrete, each side of the plane of shear, The shearing resistance of the bar then becomes a question of its rigidity in bending. Its moment of inertia in bending varies as the fourth power of its diameter and its cross section varies as the square of the diameter. Bending in it would depend on the distance apart of the centers of the bearings on the two respective sides of the plane of section but this distance increases with the increased shearing resistance afforded by the bar so that it seems roughly that the fiber stress in the bar would increase approximately with the square of the amount of the load brought upon the section of greatest shear strain, and that it would increase with the square of the diameter of the bar and be greater when the bar is nearer the neutral axis than it would be nearer the outer part of the slab.

Thus, where the shearing stress on the section is as high as 250 or 300 pounds per square inch, a bending stress in the steel of considerable magnitude may be developed particularly in slab rods that are as large as 5/8″ diameter, buried in a slab as thin as 8″.

The fiber stress produced by flexural resistance to shear on each side of the plane of section at the edge of the cap will disappear with the removal of the load but the permanent bending at such section brought about by yielding of the concrete will remain and this residual strain which is not elastic can be allowed for by noting the lack of recovery and using the final reading under zero load as the initial point from which to reckon the elastic deformation which is a measure of the true axial stress. The difference between the initial reading and the reading under maximum load represents the sum of the elastic plus the inelastic deformation of the concrete where the steel has not been over stressed and to determine the true elastic axial stress in the steel under maximum load we must use the recovery reading as the zero from which to determine the the maximum elastic resistance exerted by the reinforcing element.

The magnitude of the error involved in many published tests in regarding the true deformation as that measured from an initial zero reading instead of using the deformation shown by the amount of recovery after loading one or more times, may not be greater than 5 to 8 percent where the stresses do not exceed 8,000 to 10,000 pounds per square inch. Where the measured fiber stresses exceed 15,000 to 25,000 pounds this correction may become as great as 25 to 30 percent of the apparent stress. Local bending of considerable magnitude is induced in the steel by the relief of shrinkage strains at and next to the edge of the column capital where its junction with the slab is at too sharp an angle. These shrinkage stresses are greatly reduced where the capital is formed in an easy curve joining the column and slab.

6. The Distribution of Moments Between Mid-Span and over the Supports in Continuous Beam and Slab Construction. In Chapter VII we have shown how the moments are divided in beams between mid span and support dependent on the restraint at the supports and the relative moments of inertia in resisting bending at mid span and at the supports. In like manner in Chapter X in sub-division of moments over the supports and at mid span has been apportioned in the column supported flat slab in accordance with the moments of inertia but this division of resistance is that of flexure and when arch action is superposed upon the distribution of moments due to pure flexure the moment of the thrust is to be deducted from the moment resisted at mid span by the steel and from moment resisted at the columns as well. We can then after having determined the approximate amount of arch action, proceed to compare the measured steel stresses with the computed applied bending moment taking into consideration the added efficiency of the steel in the areas of synclastic curvature under the principles deduced in Chapter

VIII and apply the minor corrections before discussed, when warranted by the severity of the applied test load.

7. Tests Compared on the Basis of Relative Severity. In Chapter IV we have shown that the breaking down point of a member depends with respect to the concrete on the magnitude of the angular shearing deformation and this angular shearing deformation is measured by the sharpness of bending determined by M/I in which M represents the applied bending moment in the particular part of the slab or beam and I the resisting moment of inertia of the section.

In comparing tests of the same classes of floors or beams the serverity of the test as measured by the ratio M/I must be taken into consideration since with the increase in load the modulus E_c of the concrete decreases and the structure is brought nearer to the yield point value of the coaction of the metal and concrete combination.

A severe test tends to bring out inherent weakness on the one hand or toughness on the other. It tends to bring to the attention of the engineer any defect in the proportionality of the cross section and arrangement of the reinforcing elements. While every part should be proportionately strong in different structures, it may be observed that the formula M/I is wholly different from a comparison of mere steel stresses because it is a comparison not of one element but of the coaction of the two elements. Steel stresses are determined by M/Ajd, while $M/I = M/Aijd^2$. When the bending moment increases, if E_c decreases the neutral plane moves toward the steel and i decreases more rapidly than does j. As I decreases with the decrease in E_c the angular bending per unit of applied moment increases and the severity of the test accordingly depends on the variation of the ratio M/I.

Another reason why the relative severity of the test should be given consideration is because under the sharp bending of the severe test the fiber stresses become a larger percentage of the axial stress and the axial stresses themselves increase in amount as the coefficient E_c decreases throwing the neutral plane nearer to the center of action of the axial steel stresses. Further, at sections where the maximum shear and bending stresses occur at one and the same point the steel is brought into action in resisting shear. The modulus of shearing rigidity F of the steel is from 10 to 12 times as great as F for concrete. Consequently, the steel transfers a part of the shearing stress laterally from the plane where sliding would otherwise occur into the mass of the concrete on each side of such plane. In so doing the reinforcing elements are brought into local bending and the center of bearing on each side of the plane of shear moves outward from this plane as the shearing force increases so that where the shearing stress is sufficiently great to bring the steel into effective action in resisting shear the bending stresses from shear increase very rapidly with the load and as the extreme fiber stresses only are measured by the extensometer it is important to allow for these subsidiary actions in making a proper interpretation of a severe test while they may be disregarded in the interpretation of tests of low severity.

Modifications of stress both from the curvature of flexure and the bending resistance to shear render the measured stress greater than the axial stress which furnishes the moment of resistance to balance the applied moment of the loads. Because E_c for the concrete is greater for lower intensities of loading the arch action is relatively greater, and for lower intensities of load up to generally about the working load the correction for increased arch action and for the difference between fiber stress and axial stress commonly compensate for each other, rendering the measured stress treated as axial stress and the applied bending moment in substantial agreement when arch action is allowed for as before, which would not be the case were it not for this compensation of minor corrections just discussed.

8. Effect of Steel Displacement. Whether the steel is emplaced as required by the engineer's plans or is displaced either vertically or laterally has an effect on the efficiency and overload capacity of the structure. Any vertical displacement of steel toward the neutral axis increases the steel stress under load since *jd* the lever arm of the steel is thereby reduced. The test capacity or limit of overload without injury is reduced in the more rapid ratio of ijd^2. This ratio in a continuous beam or slab is modified by the manner in which the resulting variation of I at the section of displaced metal enters into the apportionment of the moments over and between the supports. It thus appears that a vertical displacement which may not increase the working stress more than twenty percent may readily reduce the limit of substantially elastic coaction of the concrete and metal by fifty percent.

9. The Difference in Distribution of the Energy Stored by the Elastic Deformation of Reinforced Concrete Beams and Column Supported Flat Slabs Under Load Establishes a Radical Difference in Mechanical Operation.*

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*Reprinted from farm paper in Franklin Institute Journal, Oct. 1918, by H. T. Eddy.

(1) Since the most accurate and complete test data available respecting reinforced concrete beams is found in Technological Paper No. 2, U. S. Bureau of Standards,[1] in which are given details of tests upon 333 of these beams, for convenience of comparison the developments in this discussion will deal with beams reinforced and loaded in the same manner as those in this Paper No. 2.

The beams in Paper No. 2 had a span between supports of $L=12$ ft., a width of $b=8$ in., a total depth of 11 in., and a depth to the center of the bottom layer of steel of $d=10$ in., a total actual length of 13 ft., a uniform reinforcement from end to end, and equal concentrated loads $=\frac{1}{2}W$ placed at each of the one-third points of the span. The reinforcement consisted of half-inch round rods from two to eight in number, of mild steel with an elastic limit between 33,000 and 41,000 lbs., a yield point between 35,000 and 43,000 lbs., and an ultimate strength between 52,000 and 64,000 lbs.

The reader is referred to the paper itself for the numerical results of the tests, except so far as herein quoted, as well as for the details of the construction of the beams.

(2) Let M designate the bending moment, I the moment of inertia, R the radius of curvature, k the proportional depth of the neutral axis, and D the deflection at mid span. None of these quantities except D vary in value between the one-third points, and they also have nearly the same values in the end thirds for some distance from the one-third points toward the ends. Especially is this the case with I and k, as may be known from Figs. 22 to 30, inclusive of Paper No. 2, which reveal the fact that the cracking of the concrete by tension extends throughout almost the entire length of the beams, except for a short distance near each end, where the bending moment is so small that the variations in the value of I and k by reason of cracking have a negligible effect.

Let $A=pbd$ be the cross section of the reinforcing steel in square inches, when $p=$ the steel ratio. Let $kd=$ the depth of the neutral axis below the top of the beam and $i=1-k$; also let $jd=$ the depth of the center of action of the steel below the center of compression of the concrete. In Paper No. 2, the parabolic law of compression is assumed, *i. e.*, $j=1-\frac{3}{8}k$, whereas for straight line compression $j=1-\frac{1}{3}k$.

If we assume that the concrete is so cracked that the steel supplies the only tensile resistance, we have from the fundamental equation of flexure, $EI/R=M$, in which the applied bending moment $M=\frac{1}{6}WL$ throughout the middle third of the span,

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[1] By Humphrey and Losse, Government Printing Office, Washington, 1912.

Hence,

$$EI=\tfrac{1}{6}WLR \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

But by the similar triangles of the distortion of bending

$$e_s : id :: 1 : R, \text{ or } R=id/e_s=Eid/f_s \ldots\ldots(2)$$

in which $e_s=f_s/E$ is the unit elongation of the steel, E its modulus of elasticity, and f_s the unit stress.

Hence,

$$EI=MR=\frac{id}{6e_s}WL \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

By the usual methods of finding the deflection of beams from their curves of flexure we find the value of the deflection D' at a one-third point of the span from the expression

$$EID'=\frac{WL^3}{432} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

and for the total deflection D at mid span

$$EID=\frac{23\ WL^3}{1296} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Hence,

$$D-D'=0.87D \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

or each one-third point moves downward in bending 87 per cent of the total deflection D at mid span.

Now substitute the value of EI given in (3) in (5) and we have

$$D=\frac{23\ L^2e_s}{216\ id} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

in which it is tacitly assumed that the mean value of I used in integration in finding the beam deflection in (5) does not differ from the value of I in the middle third as used in (3) by more than a negligible amount. This assumption has been verified as practically admissible by a more complete analysis based on the exact and somewhat larger values of I that occur near the ends of the beams due to concrete and steel acting together, with the resulting larger values of k. It is found that the effect of this upon the ratio 0.87 in (6) may be safely neglected, since although D-D' and D may be changed by small amounts their ratio will be affected by only an inappreciable amount.

Now the work performed by the load and expended upon the beam in the process of gradually loading it is expressed as follows:

$$U=\tfrac{1}{2}W(D-D')=0.435WD\ldots\ldots\ldots\ldots(8)$$

in which $D-D'$ is the vertical distance moved through by the load in performing its work. Consequently by (7)

$$U=\frac{10\ WL^2e_s}{216\ id}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Again, the work stored in the steel of the middle third of the span may be written

$$U_s=\tfrac{1}{6}\,ALf_se_s\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

in which $\tfrac{1}{3}AL$ is the volume of the steel and e_s its unit elongation. But since the moment of resistance of the steel is equal to the moment of the applied forces we have

$$Af_s\,jd=\tfrac{1}{6}WL\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

Therefore,

$$U_s=\frac{WL^2e_s}{36\ jd}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

Hence,

$$\frac{U_s}{U}=\frac{216\ i}{360\,j}=0.6\frac{i}{j}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

In case we assume provisionally that $k=0.4$, which is a good average value for the beam tests in Paper No. 2, we have $i=0.6$ and $j=1-\tfrac{3}{8}\,k=0.85$. Then $U_s/U=0.4235$ is the fraction of the total work of the load which is stored in the steel in the middle third.

It will be fair to estimate the work stored in the steel in the two end thirds as one-half of this amount, because were the same relation to exist in the end thirds between the applied bending moment and the resistance of the steel as in the middle third, then the energy stored in the steel in each end third, and due to a moment uniformly decreasing to the end, would be one-third that in the middle third. But, owing to the existence of tensile resistance in the concrete near the ends, the stresses on the steel are thereby reduced somewhat below one-third, and the amount of work expended upon the steel in each end third may reasonably be taken as reduced to not more than one-fourth that in the middle third or possibly to even less than that, thus giving as an estimated ratio of the total work stored in the steel of the entire span to the work of the load the quantity

$$\frac{3}{2}\frac{U_s}{U}=0.9\frac{i}{j}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

or in case $k=0.4$, we have

$$\frac{3}{2}\frac{U_s}{U}=0.635$$

This is considerably more than one-half of the total work of the load, being some five-eighths of it.

(3) The work U_c expended in compressing the concrete of the middle third and stored in it during loading may be calculated in like manner. For, on the assumption that beam sections originally plane remain plane after flexure we have

$$\frac{id}{e_s}=\frac{kd}{e_c}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

Hence by (7)

$$D=\frac{23L^2e_c}{216\ kd}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

But U_c is half the product of the total amount of the reduction of length of the concrete at the center of application of the compressive force multiplied by the total compressive force, *i. e.*, in case of parabolic compression.

$$U_c=\frac{1}{2}\times\frac{1}{3}L\frac{5e_c}{8}\times\frac{2}{3}f_cbkd=\frac{5}{72}\ Lbkdf_ce_c\ldots.(17)$$

in which $\frac{5}{8}e_c$ is the unit compression at the centre of gravity of the parabolic area, and the factor following is the total compressive force above the neutral axis.

But since the resisting and the applied moments are equal,

$$\frac{2}{3}f_c\,bkdjd=\frac{1}{6}WL,\text{or }bkdf_c=\frac{WL}{4\,jd}\ \ldots\ldots\ldots.(18)$$

Hence,

$$U_c=\frac{5WL^2e_c}{228\,jd}=0.022\,\frac{WL^2e_c}{jd}\ \ldots\ldots\ldots\ldots.(19)$$

Now using (8), (16) and (19) we have

$$\frac{U_c}{U}=0.475\,\frac{k}{j},\ \text{or}\ \frac{3}{2}\,\frac{U_c}{U}=0.712\,\frac{k}{j}\ \ldots\ldots\ldots\ldots(20)$$

In case we use $k=0.4$ we have for parabolic compression, $U_c/U=0.223$ as the fraction of the work stored in the concrete in the middle third. If this be increased by 50 per cent for the two end thirds we have

$$\frac{3}{2}\,\frac{U_c}{U}=0.335\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21)$$

for the total compression in the concrete. This last estimate in (21) may be rather excessive as appears when we take the sum of (14) and (21), making a total of

$$\frac{3}{2}\,\frac{(U_s+U_c)}{U}=0.97\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

since it would seem probable that more than 3 per cent might be lost in cracking of concrete and slipping of rods in concrete. For this case we have

$$\frac{U_s}{U_c}=1.226\,\frac{i}{k}=1.9$$

However, (21) is somewhat modified in case we assume straight line compression, when we compute instead of (17):

$$U_c=\frac{1}{2}\times\frac{1}{3}\,L\times\frac{2}{3}\,e_c\times\frac{1}{2}\,f_cbkd=\frac{Lbkdf_ce_c}{18}\ \ldots\ (23)$$

But by equating the resisting and applied moments we have

$$\tfrac{1}{2}f_cbkdjd=\tfrac{1}{6}WL,\ \text{or}\ b\,k\,d\,f_c=\frac{WL}{3\,jd}$$

Hence,

$$U_c=\frac{WL^2e_c}{54\,jd}=0.0185\,\frac{WL^2e_c}{jd}\ \ldots\ldots\ldots\ldots\ldots(24)$$

and

$$\frac{U_c}{U}=0.4\,\frac{k}{j}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

Now if $k=0.4$, and $j=1-\frac{1}{3}\,k=0.87$ we find $U_c/U=0.184$,

and $\frac{U_s}{U}=0.414$, and $\frac{3}{2}\,\frac{U_s+U_c}{U}=0.897$, a not improbable value.

For this case we have

$$\frac{U_s}{U_c}=1.5\frac{i}{k}=2.25$$

The correct values seem probably to lie between those calculated from the hypothesis of parabolic compression and those calculated from the straight line hypothesis. The results thus arrived at indicate that 60 per cent or more of the work expended upon the beam during flexure by the downward movement of the load is stored as potential energy of elastic elongation in the steel, and that approximately half as much has been expended in the longitudinal compression of the concrete.

These results are confirmed by the results of calculations detailed in the accompanying tables, which are based on the average values of k obtained in the tests of beams of seven different steel ratios, as given in Table 26, page 60, of Paper No. 2.

Average Work in Reinforced Concrete Beams as Per Tests in Table 26, Page 60, U. S. Technological Paper No. 2.

	No. of ½ inch rods	k, mean of 3 beams	$j=1-\frac{3}{8}k$	$\frac{3}{2}\frac{U_s}{U}=0.9\frac{i}{j}$	$\frac{3}{2}\frac{U_c}{U}=0.7\frac{k}{j}$	$\frac{3}{2}\frac{(U_s+U_c)}{U}$	$\frac{U_s}{U_c}$	$j=1-\frac{1}{3}k$	$\frac{3}{2}\frac{U_s}{U}=0.9\frac{i}{j}$	$\frac{3}{2}\frac{U_c}{U}=0.7\frac{k}{j}$	$3\frac{(U_s+U_c)}{2U}$	$\frac{U_s}{U_c}$
f_s=16,000 #	2	.358	.866	.667	.294	.961	2.27	.881	.656	.290	.946	2.26
	3	.380	.8575	.651	.309	.960	2.16	.873	.639	.311	.950	2.05
	4	.418	.843	.621	.351	.972	1.77	.861	.608	.345	.953	1.76
	5	.407	.847	.630	.342	.972	1.84	.864	.618	.335	.953	1.85
	6	.428	.840	.615	.363	.978	1.69	.857	.600	.356	.956	1.68
	7	.443	.834	.602	.378	.979	1.59	.852	.587	.371	.958	1.58
	8	.455	.829	.591	.382	.982	1.51	.848	.578	.383	.961	1.51
	Mean	.413	.845	.625	.360	.975	1.81	.862	.614	.341	.955	1.81
f_s=32,000 #	2	.288	.892	.719	.230	.949	3.13	.908	.705	.225	.930	3.13
	3	.325	.878	.692	.261	.953	2.65	.892	.681	.258	.939	2.64
	4	.360	.835	.690	.306	.996	2.25	.880	.654	.291	.945	2.25
	5	.359	.865	.666	.296	.962	2.25	.880	.656	.290	.946	2.26
	6	3.81	.857	.650	.317	.967	.205	.873	.638	.311	.948	2.05
	7	.402	.842	.641	.339	.980	1.90	.866	.621	.330	.951	.188
	8	.416	.844	.623	.351	.954	1.77	.861	.611	.345	.956	1.77
	Mean	.361	.859	.669	.300	.969	2.29	.880	.652	.293	.945	2.22

(4) In order to determine what part of the total work done by the load on a flat-slab floor is expended upon and stored in the slab steel reinforcement, assume that the center of gravity of a uniformly

distributed panel load W moves downward during loading through a distance equal to one-half of the total deflection D_2 of the panel centre. This estimate of a mean deflection $= \frac{1}{2}D_2$ probably exceeds the actual amount of the mean downward motion by a very small amount in case of many panels equally loaded, but is probably somewhat too small for the case of a single-loaded panel, especially for a deck slab, or one supported on columns of small stiffness, for, under such conditions the lines of inflection approach nearer to the supports, and the dish-shaped or suspended spans around the panel centers are some-what larger in area than when all panels are loaded. This assumed value of $\frac{1}{2}D_2$ is, however, sufficiently accurate to enable us to make a satisfactory comparison of work done in slabs with that already found for beams.

The expression for the work of a load gradually applied to a panel is one-half the product of the load W, by the mean deflection $\frac{1}{2}D_2$ *i. e.*,

$$U = \tfrac{1}{4}WD_2 \qquad \text{(26)}$$

in which D_2 may be obtained by test, or usually with sufficient accuracy from equation (71) page 204, "Concrete Steel Construction," Eddy and Turner edition of 1914. Now making use of this value of D_2 first, before we have recourse to experimental confirmation,

$$U = \frac{W^2L^3}{19\times10^{10}\,A_1\,d^2} \qquad \text{(27)}$$

in which L is the span in inches, A_1 the cross section of a single belt in case of four-way reinforcement of side and diagonal belts, and d is the depth of the center of gravity of the steel below the top of the slab at the panel center.

Again, U_2 the work stored in the slab rods of a panel is equal to one-half the product of the volume of steel in the rods, multiplied by the mean value of the product of unit stress f and unit strain e, *i. e.*, by the mean value of $f_e = f^2/E$. Now the volume of steel in the side diagonal belts of a square panel is

$$v = 2\,(1 + \sqrt{2})\,LA_1 = 4.83\,LA_1 \qquad \text{(28)}$$

and the mean value of f^2 in a rod is $\frac{1}{3}\,f_s{}^2$ when f_s is the maximum steel stress at mid span, provided the stress diminishes uniformly to zero at the lines of inflection. Although it does not diminish at so rapid a rate as this near its maximum value, nevertheless $\frac{1}{3}\,f_s{}^2$

is a suitable value to assume because not all the rods in a belt have stresses at mid span as large as the largest observed in either rod.

Let us assume from Equation (34), page 183, and from equation (52), page 196, "Concrete Steel Construction," that a mean maximum value at mid span is

$$f_s = \frac{Wl}{200 A_1 d} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (29)$$

Then the work stored in the slab rods of a panel is

$$U_2 = \tfrac{1}{2} v \times \tfrac{1}{3} \frac{f_s^2}{E} = \frac{4.83\, l\, A_1}{6 \times 3 \times 10^7} \left(\frac{Wl}{200\, d\, A_1} \right)^2 \dots (30)$$

Hence by (26) and (30)

$$\frac{U_2}{U} = 0.127 = \tfrac{1}{8} \text{ nearly} \dots\dots\dots\dots\dots\dots (31)$$

By (26), (28) and (30) this ratio may also be found otherwise in terms of observed quantities without assuming (29) as follows:

$$\frac{U_2}{U} = \frac{1.07\ 1\ A_1 f_s}{10^7\ W\ D_2} \quad \dots\dots\dots\dots\dots\dots\dots (32)$$

which may therefore be used to calculate the proportion of the total work of deflection stored in the steel from test data alone and independently of theory, since it depends on observed quantities only.

For example, in the test of the Deere and Webber Building* assuming a mean observed value of $f_s = 9000$ at mid span

$$\frac{U_2}{U} = \frac{1.07 \times 226 \times 12 \times 0.15\ (9\ 000)^2}{107 \times 124\ 678 \times 0.224} = 0.126$$

Similarly in the test of the Northwestern Glass Company Building† $U_2/U = 0.121$, and in the St. Paul Bread Company Building‡ $U_2/U = 0.128$. These results of tests differ very little from the theoretical value obtained above in (31). They show conclusively that a much smaller proportion of the total work of the load is stored in the reinforcing rods of a slab than in those of a beam, in fact under test conditions only about one-fifth as much.

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‡ See "Concrete-Steel Construction," page 228 Edition 1919.
† Page 227.
* Page 232.

This difference is so large as to make it certain that beam theory is entirely inapplicable to flat slabs, as it evidently should be when we consider the physical and mechanical dissimilarity of the two structures, one of the most noticeable of which is the almost total loss of resistance of concrete in the tensile regions of beams under heavy load, as shown by the numerous cracks below the neutral axis already referred to, while ordinary tests of slabs show little if any cracking that would materially reduce the resistance of the concrete in opposing the very complicated stresses to which the concrete is subjected in the tensile zones of the panel, to all of which the concrete opposes an effective resistance, as is evident from the fact that it continues intact. The dissemination of the steel in a fine-grained structure throughout these zones enables it to bring all the concrete into action in such a way as to afford the resistance needed to effect a result such as is impossible in beams, since in the tensile zones of slabs it is subjected to a combination of several compressions, twists and shears, which it is abundantly able to bear; and it must be remembered, in this connection, that any kind of resistance called into play in such a structure assists by just so much in transmitting the load to the supports. The concrete in a slab, without being taxed beyond its direct tensile resistance, as it is taxed in beams, is able to afford at the same time other effective resistances that relieve the steel and greatly enhance the load carrying capacity of the slab.

If only about $\frac{1}{8}U$ is stored in the slab steel, where, it may be asked, is the remaining $\frac{7}{8}U$ to be found? Most of it must be found somewhere in the concrete, which is subjected to various kinds of stress not found in beams. In fact, the numerous observable and microscopic cracks in beams almost entirely prevent any storage of energy in the concrete, except that of compression, and a small amount in vertical and horizontal shear, while the almost total absence of cracks in slabs permits the storage of more energy in shears than in beams, and permits the storage of energy in the slab in twisting and in several other ways, as well as in bond shear deformations throughout the tensile zones of the slab.

The enormous difference which we have found above between the storage of energy in beams and slabs thus proves conclusively that the mode of action of slabs is entirely different from that of beams, and that their analysis must also be entirely different. It is consequently wholly inadmissible to attempt to compute slabs by beam theory.

Having thus conclusively established a radically different distribution of storage of energy between the steel and the concrete in keeping with the foregoing divergent mathematical analyses of column supported flat slabs and beams, we will now proceed to demonstrate the applicability and accuracy of the analyses heretofore derived, in detail, by comparing the moments of resistance figured from measured steel stress under load, with the computed moments of the applied loads; the theoretical position of the points of inflection with those observed under test, the relative efficiency of different relative widths of mats of crossed rods which determine the areas of circumferential resistance, and finally a checking up of the entire elastic deportment by the harmonious agreement of measured and computed deflections within the range of the elastic resistance of the slabs. This will be followed by observations and suggestions for the proper interpretation of excessive tests and a short discussion of the indication of abnormal shrinkage strains, and the causes which produce them.

For example, we might take the case of a slab simply supported on two sides in which the drawings call for the steel to be placed at 8″ from the center of the steel to the top of the slab. Suppose this steel has been raised by the workmen 1″ higher than it should be so that the depth from the center of the steel to the top of the slab is 7″ in place of 8″ and suppose that in its proper position the percentage of steel is .5 of one percent. The lever arm j would be reduced by approximately 13% while I the moment of inertia would be reduced 26%. In other words, under working loads the unit stress is 13% higher than computed but the test capacity is 26% less for the elastic coaction of the metal and concrete than before. Precise vertical emplacement is therefore important since the test capacity is reduced by vertical displacement more rapidly than the ratio of the squares of the depth of the center of steel to the outer fiber of the concrete for the steel in its correct and displaced positions respectively.

General rather than precise lateral emplacement is important in floors of Types III and IV since the efficiency of the mat of rods depends upon its covering the areas of synclastic curvature and not upon the precise lateral spacing of the bars within an inch or so of the figured lateral arrangement.

Ultimate Strength and test capacity do not follow the same law. Test capacity is the magnitude of the load which the member can support in a substantially elastic manner by the coaction of the metal and concrete without undue permanent deflection and is proportional to the ratio M/ijd^2. The ultimate strength when it depends on the steel element, varies on the other hand as M/ijd so that the ultimate carrying capacity is fortunatly not reduced so rapidly as the elastic test capacity by displacement of the metal.

10. Foregoing Analysis Supported by Published Tests of Column Supported Flat Slab Floors.

(1) **Franks Building.** Slab 9¼″ thick, panels 20′3″×19′4,″ drop head 6′0″×6′0″×4″ deep. Column cap 44″ diam. Test published Proc. Nat. Assoc. Cement Users, 1912. See also Eng. News April 18, 1912. Test was made on the 10th floor; column below, 20″ diam. reinforced by eight 1¼″ round verticals with spiral; column above, 15″ diam. six 1″ verticals.

Reinforcement, four way, 18 rods 1/2″ round in diagonal and long direct bands, 16 rods 1/2″ round in short direct bands. Spider of eight 1″ radial arms and outer 8′0″ square carrying frame of 3/4″ round. Lap of diagonals 6′0″ over the columns regarded as .8 efficient. Lap of direct bands 2′6″ over columns. Width of belts 8′0″, which is less than 7/16 ths of the mean of the side spans. Because of the relatively light load on the slab it may be considered as fully restrained, or $dz/dx=0$ at the columns, and the moments of inertia to resist bending at mid span and supports are:

$$
\begin{array}{lr}
I_0=3.5\times 8.25^2\times .90\times .71\times 3.5 & =745 \\
I_x=3.5\times 11^2\times .88\times .76\times 7.5 & 2007 \\
\qquad\qquad\text{Spider, approx.} & \underline{200} \\
\qquad\qquad\qquad\text{Total} & =2207
\end{array}
$$

$$\therefore I_x/I_0=3+\text{ nearly.}$$

The variation of the moment ratios m_0/WL or m_x/WL is less than half as rapid as that of the ratio of I_x/I_0, a circumstance which permits closely approximate calculations of these moment ratios without material error by the use of roughly approximate values of I_x/I_0.

Four panels were tested with pig iron to a load of 312 lbs. per sq. ft. The estimated dead load was 115 lbs. per sq. ft. The published values of stresses are not those measured but are those measured increased in proportion to the dead weight and a proportionate reduction has been made in their use.

Total live load per panel = 118,000 lbs.

Total applied moment = 118,000 × .062 × 19⅓ × 12 = 1,690,000 lb. ins.

The percentage of arch action in this floor is to be calculated in the manner explained in Section 3 of the present Chapter, as follows:

The mean span between the edges of the caps is 237.5–44 = 192.5″ and this is 14.6 times the thickness, 13.25,″ at the edge of the cap, hence

$$\left(\frac{1}{11}\right)^2 : \left(\frac{1}{14.6}\right)^2 :: 1/6 : .0946$$

or the arch action is 9.46 percent of the total applied bending moment for $E_c = 2{,}500{,}000$. But E_c for the concrete was found nearly 4,000,000 and is here taken as 3,750,000, *i. e.* archaction equals $1{,}690{,}000 \times .0946 \times 1.33 = 213{,}000$. The total applied moment is distributed between supports and mid span in accordance with the principles and tables already developed and is to be corrected for arch action as follows:

$m_x = 1{,}260{,}000 - 213{,}000$ (arch action) = 1,047,000 lbs. ins.
$m_o = 430{,}000 - 213{,}000$ (" ") = 217,000 lbs. ins.

Hence the sum total of the applied moments computed from the loads and corrected for arch action = $1{,}047{,}000 + 217{,}000 = m_x + m_o = 1{,}264{,}000$ lbs. ins.

Resisting moment m_o	$= 3.5 \times 11 \times .88 \times 5 \times 3180 = 538{,}000$
over the center	$3.5 \times 11 \times .88 \times 2.5 \times 3540 = 302{,}000$
column at 312 lbs.	
per sq. ft.	$2 \times 9 \times .8 \times 2000$ (Rad rods) $= 45{,}000$
Cross band lap and	
ring estimate	$3.5 \times 11 \times .88 \times 2200 \times 74{,}000$
	959,000 lb.

Resisting moment m_x	$= 3.5 \times 8.25 \times .9 \times 4500 = 117{,}000$
	$3.5 \times 8.25 \times .9 \times 1400 \times 2.5 = 91{,}000$
	208,000
	$m_o + m_x = \dots\dots$ 1,167,000lb.

Half the sum of the resisting moments over the supports, plus the moment at mid span equals 1,167,000 and that computed from the loads is 1,264,000 which is a fair agreement. The true stresses are here somewhat reduced by initial shrinkage stresses. The effect of such stresses is less than in the simple beam because they are induced both in the top and bottom of the floor instead of in the bottom only as in the simple beam.

After the loading of 312 pounds over the four panels, which load was too light to stress the steel to such an extent that the effect of the knee in the stress strain of the steel caused by shrinkage stresses would be entirely eliminated, two panels situated diagonally were loaded by doubling up the load. Under this applied load of 624 lb. per sq. ft. the total applied moment will be twice that computed for 312 lbs. viz. 3,380,000 lb. ins. of which the moment due to arch action is $3,380,000 \times .0946 = 320,000$ provided we take $E_c = 2,500,000$ in this case of higher stresses in the concrete by reason of the greater load. Since this moment of arch action is to be deducted from the moment of the steel stresses both at mid span and supports, the sum total of moments of steel at mid span and half those over the supports will be $3,380,000 - (2 \times 320,000) = 274000$. Under this applied load of 624 lbs. per sq. ft. after including an increase for 115 lbs. dead load the average unit stress on the diagonal was 9280 lbs. on the cross band, 8140 lb. over the central column and 5315 lbs. over the side column. We therefore have to compute the side and center column separately each on a different basis in which the reported stresses have been reduced in the ratio of 739 to 624 in order to obtain actual stresses.

Resisting Moment	$m_x = 3.5\times 11\times .88\times 5 \times 7800 =$	1,330,000
over center column	$3.5\times 10\times .88\times 2.5\times 6850 =$	525,000
computed from measured	Rad. rod and short lap estimated	206,000
steel stresses		2,051,000

Resisting Moment at side column computed for measured
cross band stress 4400 lbs. and assumed diagonal band stress 7800 lbs.
same as center column $m_x = 1,862,000$

Resisting Moment	$m_x = 3.5\times 8.25\times .9\times 1.5\times 8,500 =$	336,000
at mid span from	$3.5\times 8.25\times .9\times 2.5\times 5,850 =$	450,000
measured stress		786,000

Hence half the sum of the resisting moments over the supports, plus those at mid span, figured from measured stresses as noted is 2,742,000 lb. ins. But half the sum of the moments over the supports, plus that at mid span, computed from the loads = 2,740,000 lb. ins., thus showing a better agreement than previously because of the higher intensity of loading.

(2) **Larkin Building.** Test published in Proc. Natl. Assoc. Cement Users, 1913. Cement Era, Jan. 1913; Eng. Contracting Feb. 1913. A four way flat slab, panels 24′2″×20′2″, 9″ thick in

the rough. Column cap 60″ diam., drop head 8′0″×8′0″×6¾″. Five panels were uniformly loaded throughout. Spider, eight radial rods 7/8″ round, carrying rods 5/8″ and 1/2″ round. Reinforcement of interior panels, diagonal bands 19 rods 1/2″ round 35′0″ long and 8′9″ wide; long direct band 7′9″ wide of twenty-three rods 1/2″ rounds; short band 8′9″ wide of thirteen 1/2″ rounds; wall panel 26 rods 1/2″ round in long direct band, diagonals as in interior panels. Laps were meager, amounting to only 1′8″ over the columns in diagonal bands, and 1′10″ to 2′0″ on long direct bands, and on short band, half 2′0″ laps and half 7′0″ laps, the latter only being fully effective in mat action. This statement as to efficient operation is verified by the table of comparative stresses given in Cement Era, January 1913, p. 56, which is quoted below:

TABLE OF COMPARATIVE STRESSES OVER EDGE OF CAPITAL TOTAL DEAD AND LIVE LOAD IN FLOOR

Loads	370	535	738
	Design load		
Short cross band	1,710	2,760	6,540
Diagonal band	3,620	5,600	7,880
Long cross band	3,000	3,540	7,000

TABLE OF COMPARITIVE STRESSES OVER EDGE OF DEPRESSED HEAD TOTAL DEAD AND LIVE LOAD ON FLOOR

Loads	370	535	738
	Design load		
Short cross band	2,000	2,760	5,500
Diagonal band	5,120	8,270	13,470
Long cross band	4,270	9,300	13,900

Because the ends of the laps in the diagonal bands extended only to the edge of the drop the resistance of the laps was radial only, otherwise the stresses over the edge of the drop could not have been as great as over the edge of the cap, but the fact that they were nearly *twice* as great confirms the explanation here proposed.

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*See page 57 Cement Era, January 1913.

Taking the long direction of the panel and figuring the laps and light spider as adding 25% to the moment of inertia over the head, we have the long way of the panel–

$$I_x = 2520 \text{ in.}^2 \qquad I_0 = 569 \text{ in.}^2$$
$$I_x/I_0 = 4.5 \text{ approximately}$$

Panel load at 535 pounds per square foot = 256,000 pounds.

Taking the diameter of the cap as $C = .2L$ the total applied moment is 4,340,000 lbs. ins., and using either the coefficients in Fig. 122 or those given in Table Page 284, Chapter 10, we have

$$\mathrm{m}_0 = 830{,}000 \text{ lb. ins.}$$
$$\mathrm{m}_x = 3{,}510{,}000 \text{ lb. ins.}$$

Percent of arch action computed as before = 11.7 percent

Moment of arch action = 4,340,000 × .117 = 510,000 lb. ins.

Hence the calculated applied moments as shown by steel stresses corrected for arch action will be as follows:

$$\mathrm{m}_0 = 830{,}000 - 510{,}000 = 320{,}000 \text{ lb. ins.}$$
$$\mathrm{m}_x = 3{,}510{,}000 - 510{,}000 = 3{,}000{,}000 \text{ lb. ins.}$$

which results are to be compared with the resisting moments shown by the measured stresses in the steel as follows:

$$\begin{array}{ll} \text{Resisting Moment} & \left\{ \begin{array}{l} 3.73 \times 8\tfrac{1}{2} \times .89 \times 2.5 \times 3240 = 173{,}000 \\ 4.5 \times 8\tfrac{1}{4} \times .9 \times 2.5 \times 2740 = \underline{93{,}000} \end{array} \right. \\ \text{at center of span} & \qquad\qquad\qquad\qquad\quad \text{Total} \quad 266{,}000 \end{array}$$

which is fair agreement with m_0 considering the low intensity of the stresses involved.

$$\begin{array}{ll} \text{Resisting Moment} & \left\{ \begin{array}{l} 3.73 \times 14 \times .87 \times 5 \times 5600 = 1{,}640{,}000 \\ 4.5 \times 14 \times .87 \times 5 \times 3540 = 1{,}250{,}000 \end{array} \right. \\ \text{over the support} & \qquad\qquad\quad \text{Spider} \qquad\quad \underline{112{,}000} \\ & \qquad\qquad\qquad\quad \text{Total} \quad 3{,}002{,}000 \end{array}$$

which is in good agreement with 3,000,000 computed from the loads.

In computing the above moment of resistance the short laps are taken as effective circularly at the edge of the column cap but as inefficient between the edge of the cap and the drop head as appears from the measured stresses shown on the preceding page. The narrow relative width of the diagonal belts renders these belts only about eighty percent efficient.

In the preceding analysis of the applied bending moments and moment of inertia the diagonal of the panel was taken as the axis of zero shear. When a drop head is of the same rectangular form as the panel as treated in Fig. 123, Chapter 10, this is exact. Otherwise not.

The drop head in the Larkin building is square and rigid because of its relatively great thickness. This causes the axis of zero shear to coincide with the diagonals of the drop head within the head and to follow the line from diagonal corner to diagonal corner of the drop heads along the diagonals of the panel. Thus the centers of the diagonal bands are unsymmetrically situated with reference to the diagonal axis of zero shear, and the twisting moments and bending moments are changed slightly thereby. The principal effect of this, however, consists in a modification of the moment of inertia at the supports in the short cross direction since the center of the diagonal belt is not over the corner of the drop but to one side, strengthening the longer side and weakening the shorter side of the column head.

Taking this fact into consideration the ratio of I_x to I_0 for the short side is approximately four, while lengthwise the increase is relatively smaller taking care of the increased moment in the long direction by this shift of the axis.

On the basis of this ratio and using the coefficients obtained in the table already employed the total applied moment is

$$\mathrm{m}_0 + \mathrm{m}_x = .054 \times 256{,}000 \times 240 = 3{,}300{,}000$$

and the correction for arch action is

$$3{,}300{,}000 \times .117 = 388{,}000$$

while, $\mathrm{m}_0 = .0237 \times 256{,}000/2 \times 240 = 730{,}000$
and $730{,}000 - 388{,}000 = 342{,}000$ lb. ins., is the applied moment at mid span to produce steel stresses in the short direction.

$$\begin{array}{lll} \text{Resisting Moment} & 13 \times .196 \times 8\tfrac{1}{4} \times .91 \times 12{,}300 = & 236{,}000 \\ \text{at mid span} & 19 \times .196 \times 8 \ \times .89 \times \ 3{,}240 \times 2.0 = & \underline{171{,}000} \\ & & 407{,}000 \text{lb.ins.} \end{array}$$

This moment is somewhat greater than the applied moment figured from the loads, assuming $dz/dx = 0$ at opposite columns, but there would be some slight tipping of the columns under the condition of three panels loaded which would tend to increase this monent slightly but because the columns are large and relatively very stiff the major portion of this increase is to be looked for in the lack of uniformity of the reinforcement of the plate. Thus in the cross band there is only 13 rods against 23 in the long direct band, presenting a wide variation between $5\frac{1}{4}''$ and $8\frac{3}{4}''$ center to center of the bars in the respective belts, an arrangement which would cause relatively high stresses in the cross band, such as indicates an unscientific distribution of metal.

Point of Inflection. This test was conducted particularly to locate the points of inflection. The columns were stiff, so that the restraint was nearly complete at the column supports. Hence the points of inflection will be determined with practical accuracy from the table Chapter X, for the relative moments of inertia over the supports and between the supports.

On the long side of the panel since the ratio of I_x to I_0 is 4.5 the point of inflection for pure flexure along the long side is 8′3″ distant from the column but the arch action produces an opposite bending moment of 510,000 lb. ins., which renders the state of stress nearly constant for a considerable distance inside of the line of inflection. This distance can be determined by laying off the moment produced by the arch thrust on the moment diagram and scaling the horizontal distance through which the negative moment is neutralized by this opposing moment, which is found to be 11½″. Subtracting this distance from the 8′3″ gives an apparent point of inflection 7′3½″ from the column. As determined by this test 7′3″ was the apparent point of inflection which is in close agreement with the theoretical value.

In the short direction, taking the ratio of I_x/I_0 as 4, the theoretical point of inflection is 6.7′ from the column and the moment of arch thrust 388,000 in. lbs., would move this back .7′ giving the apparent point of inflection as 6′ which agrees with the value determined by test.

Moments Figured in the Diagonal Direction of the Panel. In the preceding computation the bending moments have been figured normal to the sides of the panel. By aid of the axes of zero shear which are the median lines of the panel parallel to the side, we may also figure the bending moment applied normal to the diagonal and again check the stresses in this direction.

The median lines of the panel and the panel edges or direct column lines divide the panel area into four squares. Each of these lines being an axis of zero shear we may treat the load of each of these squares separately so far as deriving the bending moment on the diagonal is concerned. Thus each quarter panel has a moment about the column that supports its load. The center of gravity of this area is at the quarter point of the diagonal. Hence the bending moment diagonally may be taken as the product of 1/4th of the panel load multiplied by the quarter length of the diagonal less one half of the diameter of the column cap. This total applied moment is accordingly–

$$\tfrac{1}{4}\times 256{,}000\left(\tfrac{1}{4}\sqrt{20^2+24.16^2}-2.25\right)\times 12=3{,}860{,}000 \text{ lb. ins.}$$

which is closely a mean between the applied bending moment in the two respective directions but it is more eaxctly the product of a coefficient for the mean span multiplied by the load times the mean span $=\dfrac{a^2+b^2}{a+b}$, in which a and b are the long and short sides of the panel respectively. Now this bending moment is subdivided between the supports and mid span in such a way that the sum of the negative moment areas near the supports, divided by the moment of inertia at the supports is equal to the total positive moment area at mid span when divided by the moment of inertia at mid span, *i. e.*, $B/I_x=A/I_0$.

Now the moment of inertia over the supports diagonally may be taken as a mean between the moments of inertia normal to the sides or $I_x=2390$, while that at mid span will be the sum of the moments of inertia across the median lines of the panel in both directions because the metal cut by the planes of division of the diagonal beam squares pulls from the columns supporting both the diagonal beams. This gives a ratio of $I_x/I_0=2.2$, from which the point of inflection on the diagonal is determined graphically as 8′9″ and as modified by 11.7% arch action the apparent point of inflection is 7′1¼″.

This position of the line of inflection is based on the consideration of the lines of zero shear as being the true diagonal of the panel. The unsymmetrical form of the drop head, it being square instead of rectangular as the panel is, would have a tendency to move the points of inflection perhaps 2″ or 3″, while the flatness of the moment curve at this point would make it very difficult indeed to determine experimentally the apparent point of change of stress within 1″ or 2″.

The measurements indicate this apparent point of inflection to be 7′6½″ from the column center along the diagonal. The bending of the interior column at the edge of the load area might increase the distance of the point of inflection from the central column slightly, but without complete data as to the reinforcement of the columns this cannot be precisely investigated though it appears doubtful whether the total tipping of the columns would permit the measured points of inflection in the diagonal direction to agree precisely with those computed on the assumption that the floor is perfectly restrained at the column.

Approximate Estimate of the Effect of Narrowing the Diagonal Belts in Reducing Their Efficiency and in Increasing Center Deflections. Computing the deflection in the Larkin test according to formula in Section 8, Chapter X, equations (21) and (23), we find the computed deflection on the diagonal amounts to .2505 ins. The measured deflection .27, indicates that the estimate made of the efficiency of the narrow belts of 80% may be 5% too high since although the deflection would be increased somewhat by the tipping of the columns an amount which may be precisely figured were the steel reinforcement of the column known the deflection would be reduced on the other hand by the arch action by approximately 9% or to .228. Thus the amount that it is to be increased by the tipping of the columns appears from the size of the columns somewhat too great to be accounted for fully on that basis and we therefore conclude that the efficiency of the belt is about 5% less than 80% of the normal efficiency as was assumed as a basis for the foregoing computation because of the narrowness of diagonal band.

To apply the further corrections that it is possible to figure from the unsymmetrical form of the cap and the shifting of the line of zero shear from the true diagonal of the panel to a broken line at 45° from the center of the column cap to its edge and from the corner of the cap diagonally to the corner of the opposite cap in a diagonal direction would in the absence of a precise steel survey and complete data regarding column reinforcement, be carrying the computation to a degree of precision unwarranted by the completeness of the published data. Other refinements before mentioned have not been attempted for like reason.

(3) **Northwestern Glass Company Building.** Floor, a Mushroom flat slab approximately eight inches thick in the rough, interior panels 16′×17′ and wall panels about 16′ wide from the edge of the wall to the center of the opposite columns by 17′ long. Four panels were first loaded, then the load was shifted to two panels diagonally situated.

The test is described in Trans. Am. Soc. C. E. Vol. LXXVII, 1914. Basement story 10′6″, columns 30″ diameter, six 1⅛″ round verticals with 3/8″ round spiral; cap 50″. First story 16′8″, column 28″ diameter, vertical reinforcement four 1⅛″ round rods.

Slab reinforcement, four way belts of fifteen 3/8″ round rods each, width 7′4″. Spider eight 1⅛″ radial arms, rings, one 7/8″ round rod 7′3″ diameter, one ⅝″ round rod 4′9″ diameter. No slab rod splices over central column, short splices on side columns of area tested.

Loads and Equivalent Moments. The loads were applied in squares to prevent arch action in the loading, and unloaded areas were left about the columns in order to measure the deformations. The diagram of these areas is given p. 1347 and the detail of loading p. 1348 of the paper. Under load 3 the unit load on the loaded area varied from 512 to 558 lbs. per square foot. Most of the deformations were measured in panel *D*, the loaded area having a maximum unit load of 512 lbs. equivalent to 68% of this amount distributed uniformly or 350 lbs. per square foot.

Panel Load = 350 × 282 = 99,000 lbs.

Because the slab steel was lapped over one line of columns and no splices were present over the other, the moments of inertia of the columns in the line further away from the wall would be much greater than of the central line of columns. This would affect the stresses in the diagonals of panel *D* equally in each direction. The ratio of I_x to I_0 toward the center column is found to be 1.36. The same ratio for the columns with the splices is 1.84 or a mean of 1.6. $C/L = .215$. Arch action of 6% = 68,000 lb.ins.

m_0 computed from the load and the mean ratio of the moment of inertia by coefficients of the table =	356,000 in.lbs.
Arch action to be deducted	68,000
Applied moment to produce steel stresses at midspan	288,000 ″ ″
The resisting moment computed from the steel ratio and the measured stresses at mid span	267,000 ″ ″

Accounting for 92% of the applied load moment.

m_x figured from the loads is, for the mean moment of inertia,	780,000 in lbs.
	68,000
Mean applied steel moment	712,000 ″ ″

Or approximately 740,000–68,000 = 672,000 lb.ins. over the central column having the smallest moment of inertia. Computed resisting moment 687,000 lb.ins. The stresses over the column head having been given in panels adjacent to *D* which were more heavily loaded have been reduced in accordance with the increase in load so as to apply to panel *D* in which the steel stresses were measured in the bottom while corresponding measurements over the columns were not made in these directions.

It thus appears that fair agreement between theory and measured stresses has been established for loads that do not exceed substantially the range of approximately elastic coaction of the metal and concrete.

Under load 7 of 1,004 lbs. per foot of loaded area or an equivalent of somewhat over 800 lbs. uniform load, the column stresses were figured in Chapter XII, Unbalanced Loading.

It may be interesting in conclusion to compare the relative severity of this test with that of the Deere and Webber building. The moment of inertia of the Deere and Webber floor across mid span is approximately 286 compared with 202 for the Northwestern Glass Building. If we assume roughly, that the ratio of the average moments over the supports to that at mid span is the same the severity of the test would be determined substantially by M/I at mid span in the two cases. Therefore, the Northwestern Glass Company test which at the maximum load was 1004 pounds per foot on the loaded area or an equivalent of over 800 lbs. uniformly applied was 2.3 times as severe as the Deere and Webber test. Had 2.3 times the load been applied to the Deere and Webber floor it is thought that the somewhat narrower belt in the Deere and Webber floor would have given valuable data clearly confirming the theoretical deduction as to the decrease in efficiency with the decrease in width of the mat of crossed rods at mid span and over the supports.

As the Larkin building test gave valuable information showing the necessity of uniformity in distribution of the metal through the areas of synclastic curvature, whether in a rectangular or a square panel, so the test of the Northwestern Glass Company building indicates the desirability of uniformity of the arrangement of laps or splices in the floor to secure uniformity in the moment of inertia over adjacent columns if uniformity of maximum stresses in similar parts of the structure is to be secured.

As in all cases where small rods are used, kinks or sharp bending may be expected at various points. In the Northwestern Glass Company test, gage point III indicates a straightening out of one of these kinks by the lack of recovery of over 50%. This particular rod was for a slight distance uncovered at the top of the slab, facilitating such straightening out due to the lack of complete embedment. The condition of the Northwestern Glass Company slab was not particularly satisfactory. It was cast in winter time and the surface of the plastic concrete was covered with boards leaving the resulting thickness from a scant 7″ minimum to 8 and 7/16″ maximum and one of the points of minimum thickness of substantially 7″ occurred directly under gage points 110 and 111 which would tend to increase the stress in this rod.

Conclusions. From the foregoing analysis of tests of concrete floors in normal condition where the tests were not particularly severe nor carried beyond the substantially elastic coaction of the concrete and metal, it will be observed that agreement between theoretical computation of the stresses and the measured stresses is statisfactory without the necessity of supposing that the negligible tensile resistance of the concrete plays a part in the actual operation of the slab wholly inconsistent with the assumption of its negligible character on which the theoretical computations are based. No attempt has been made to apply the small corrections of fiber stress, reduction of arch action for deflection, etc., because the precision of the steel survey was not sufficiently exact to warrant such refinement although had the test loads been greatly increased such corrections would have assumed greater relative importance.

The effect of the finish coat or added thickness on the central portion of the slab for floors in which there is a wide divergence between the moment of inertia over the supports and mid span, will now be considered.

A 1¼ inch effective finish coat if added to the Northwestern Glass Company building would increase the moment of inertia of the slab at mid span by nearly 50 percent. It would increase the moment of inertia in a diagonal direction to an amount greater than that over the central support. It would greatly relieve the tension in the cantilever area, while increasing that at mid span. Thus, such a floor as that in the Northwestern Glass Company building, would be increased in strength by more than 25 percent by the addition of the finish coat while the same relation would not hold true in a floor in which there is such a wide variation between the moment of inertia over the support and at mid span as that found in the Larkin design. The moment of inertia at mid span can be increased by the addition of concrete, but that at a support cannot.

Thus, by taking proper account of the relative moments of inertia at the critical sections of the slab, as well, the arrangement of the metal vertically in the areas of synclastic curvature and its uniformity of distribution therein, it is possible to compare all published tests upon substantially the same basis, and to draw valuable conclusions therefrom dependent of course on the accuracy of the experimental work and the completeness of the test data.

No wider variation in the relative moments of inertia would probably be encountered than that found in the Larkin Building on the one hand, and the Northwestern Glass Building on the other, and any theory which can rationally predict the applied moments in agreement with measured resisting moments in two structures differing so widely cannot conceivably err in substantial particulars.

11. Interpretation of a Test Beyond the Elastic Coaction of the Metal and Concrete. *The Western Newspaper Union Building* Chicago* was an eight story reinforced concrete structure in which there were two flat slab floors. It was erected in 1909 by the Hinchcliff Company, S. N. Crowen, Archt. The floor tested was referred to somewhat without warrant in the Bulletin as a Turner Mushroom flat slab type. Sketches for this construction were made for the architect by Turner, with a typical design of the reinforcement corresponding to the load and thickness of the slab. This design was afterwards changed without consulting Turner so as to include a cross section of metal of about double that of the original Turner design and of half its efficiency. The panels of the building were 19′4½″ by 17′5½″. The slab though nominally 8½″ thick had a measured thickness from 7.51″ to 8.46″ about the central column of the loaded area and in places nearly 9″ between columns. Columns were octagonal 24″ diam. below and 21″ above the floor octagon, cap 54″ short diam.

Test Load. Four panels were loaded with pig iron as shown in Fig. 135, the maximum load reached 913 pounds per square foot.

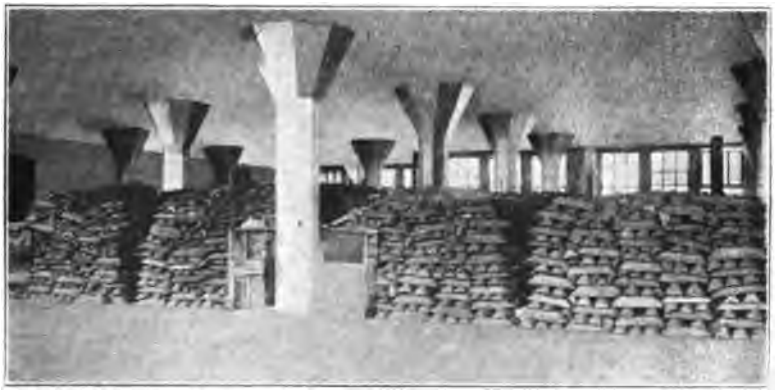

Fig.135. Test of the Western Newspaper Union Bldg.

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*Bulletin No. 106, May 1918, by Arthur N. Talbot and Harrison F. Gonnerman, pp. 48, Engineering Experiment Station, Univ. of Ill.

Form of the Capital. The form of the capital and its unscientific proportions should be particularly noted. Instead of a curved capital properly filleting the junction of the slab and the column or one which has a lateral flare sufficient to permit such moderate yielding at its outer edge as to bring the load gradually upon the center of bearing within the capital the small inclination of its sides to the vertical localize and accentuate the bending deformations directly above the edges of the cap at the top of the slab and in the concrete at the bottom. This phenomenon accords perfectly with the accentuation of bending stress indicated by theory and shown by experiment to occur in a piece of steel under flexure at the point where a sharp re-entrant angle is formed by an abrupt change in section.

Reinforcement. The reinforcement in the slab consisted of fifteen 5/8″ rods about 6¾″ between centers in four belts, a diameter of rod suitable for a 13″ or 14″ slab but wholly unsuited for an 8″ slab because of the great thickness of the mat which results when four or more layers of steel are placed one above the other. The layers of reinforcement over the columns were placed in a most unscientific manner, the rectangular belts alternating with the diagonal belts, as will be noted in the diagram showing the steel deformations in the top of the slab over column 22.

Quoting from the bulletin, p. 29:

> "The average depths of the layers of bars at the edge of the capital of column 22 ranged from 0.90 in. below the upper surface of the slab for the upper layer to 3.65 inches for the lower layer. At this column the layers of bars in the order of their position with respect to the upper surface of the slab were as follows: (1) north and south rectangular bars (one lap), (2) diagonal bars running to Northwest, (3) east and west rectangular bars, (4) diagonal bars running Northwest, (5) north and south rectangular bars, (second lap), (6) circumferential ring bars, (7) radial column bars."

From the above statement it appears that the lower layer of the bars was but little above the center of the slab, a position which would greatly reduce its relative moment of inertia in comparison to the same belt at mid span and therefore reduce the relative resistance of that layer at the support and increase the relative stress upon the same layer at mid span.

Deflection Under Various Loads. The loaded area is represented in the accompanying plan of panels *A*, *B*, *C*, and *D*, in which the deflections under various intensities of loading are shown in the plot of the deflection of the various points noted on the plan.

WESTERN NEWSPAPER UNION BUILDING
MEASURED DEFLECTIONS

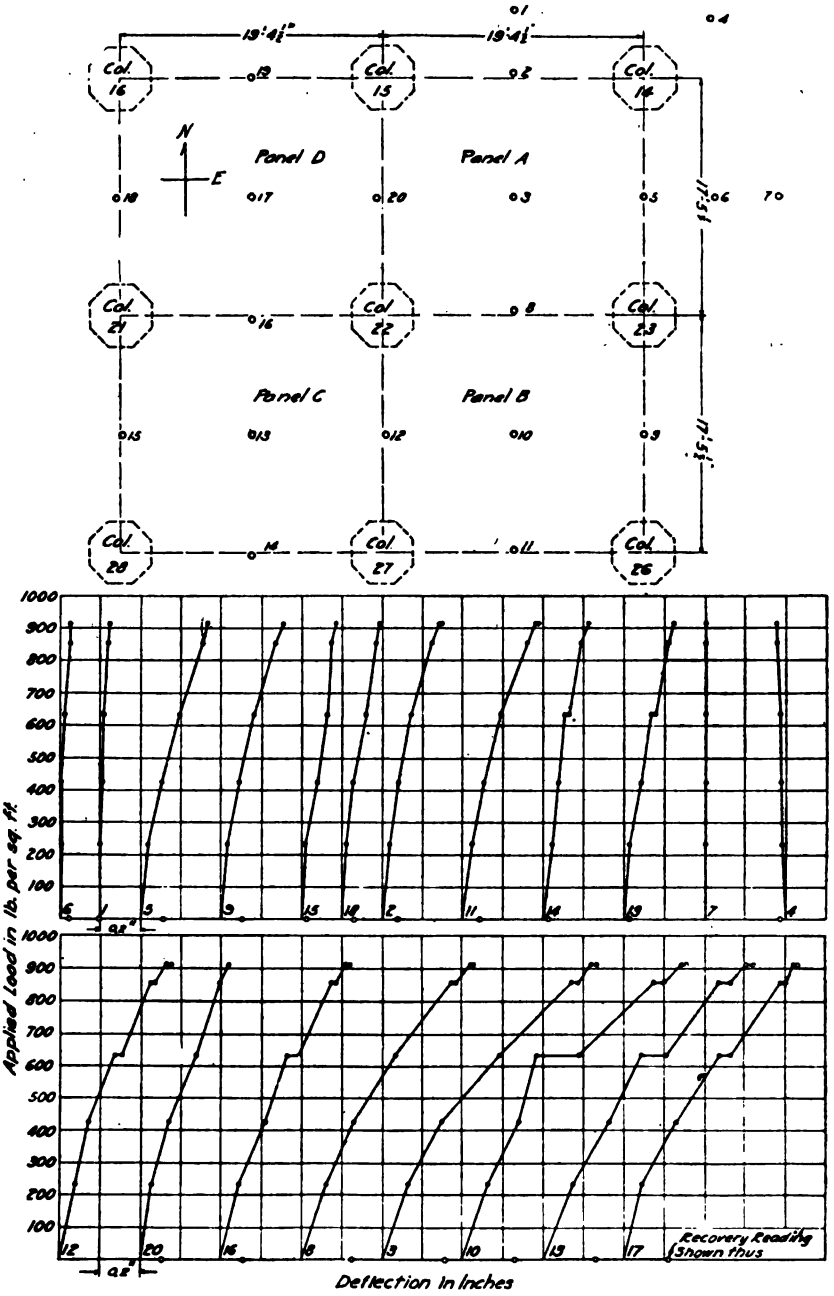

Fig. 135.

The extent of recovery or the permanent deformations are shown by the small circles indicating the recovery readings on the zero line. It will be noted that at the center of the panels the lack of recovery on points 17, 3, 13, and 10, is equal to the total deflection of the floor under the 400 lbs. applied load.

Steel deformations measured on the upper surface of the bars over the top of the cap and concrete deformations on the under side of the slab next to the cap are shown in Figs. 137 to 140, under 300, 400, 600, and 900 lbs., loading per square foot.

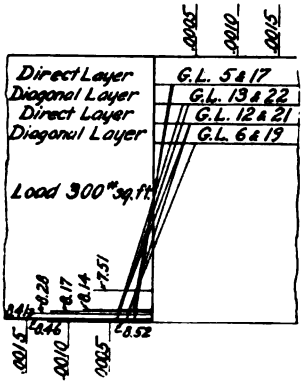

Fig. 137.

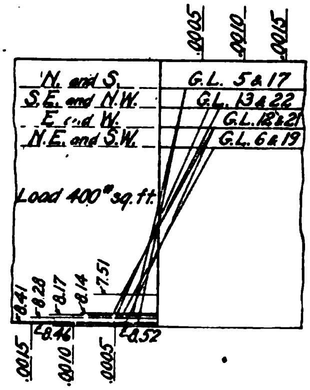

Fig. 138.

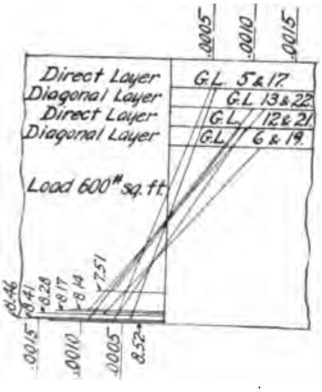

Fig. 139.

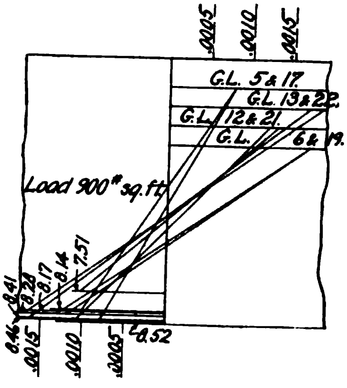

Fig. 140.

Steel Deformations at the Edge of the Slab, Column 22 and Concrete Deformations outside of the Capital at the Bottom of the Slab.

The steel deformations shown in the foregoing figures were measured upon gage lines centered over the edge of the capital while the concrete deformations below were measured at one side of that edge and were consequently less than the unit deformation at the edge because these deformations increase toward and are greatest at the edge of the cap so that to determine the true position of the neutral plane and the centroid of compression corresponding to the steel stress, the concrete measurements should be increased in amount by some twenty percent.

The steel deformations measured on the upper side of the bars increases from the outer to the inner layers. This indicates a scalloped shaped neutral surface about the column capital.

The deformation on gage lines 5 and 17 in the North and South direct belt is half that on gage lines 12 and 21 the East and West belts, beacuse in the North and South belt on which 5 and 17 were measured, there is a splice and double the cross section of metal resists the stress brought upon this belt so that the deformations are correspondingly reduced.

Comparing the measured deformations in the concrete and steel, it will be noticed that the concrete deformations increase at a more rapid ratio than the steel deformations with the increase of load; accordingly the neutral plane is moved up toward the steel somewhat more than would be indicated by the diagram because the concrete stresses are considerably higher at the edge of the cap than the average deformation measured along the gage line outside of the cap. The condition of the concrete at the edge of the cap under the 913 pound load is thus described page 27 of the bulletin:

> "The greatest strains on concrete in the lower surface of the slab were found close to the edge of the capital of column 22. The strains at this column at the diagonal gage line (See Fig. 15) ranged from 0.0012 to 0.0016 in. per in. at the maximum load. These strains are as great as the strains which were found at failure in the tests of the concrete prisms cut from the slab; they represent the range in deformation at the ultimate load usually found in compression tests of concrete. Spalling or chipping of the concrete surface was plainly visible near the edge of the capital of column 22 in panel A. At rectangular gage lines near the capital of this column a strain in the concrete of 0.0014 in. per in. was observed. These high deformations indicate that the concrete near the capital of column 22 was highly stressed and that at certain gage lines it was stressed to its ultimate strength, the action of the surrounding concrete preventing its complete failure."

Since the greatest compression in the concrete occurs at the edge of the cap and has there been stressed under the higher load to nearly its ultimate resistance as stated in the bulletin, the larger part of the permanent deformation would occur at that place. The lack of recovery shown by the deflection diagrams amounting to .3″ at mid span is the result largely of this inelastic yielding at the edge of the cap because the concrete was old, very hard and highly elastic, and because of this condition relatively little inelastic deformation could elsewhere occur.

The localization of bending at the edge of the cap would obviously result in a combination of two kinds of stress in the steel, axial stress and bending stress across the section of localized yielding, so that in determining the static moment of resistance of the steel, the fiber stress involved in the measurement of the deformation on the upper surface of the bars must be separated from the average axial stress in the total section of the metal.

Properties of the Concrete and Steel. Sample cubes cut from the floor showed a unit compressive strength of 3300 to 5400 pounds per square inch and it is not unlikely that the specimens giving the lower values may have been somewhat injured in removal from the floor.

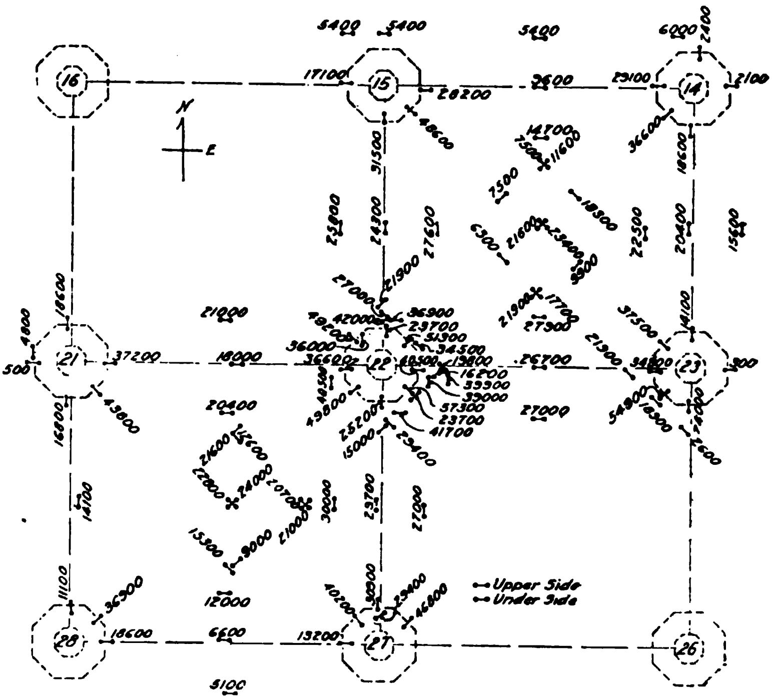

Fig. 141.

Steel Deformation, times *Es* for 913 pound loading.

Tests of the bars showed an average yield point value of 66,000 pounds per square inch, and an average ultimate strength of 100,000 pounds.

Steel Deformations. The steel deformations are given under all loads in various diagrams in the Bulletin. These deformations under 913 pound loading are multiplied by E_s the modulus for the steel and tabulated in Fig. 141.

These figures represent not the average axial stress on the bars but the approximate fiber stress on their upper surfaces where the measurements were made. Fig. 142 shows the deformations multiplied by E_s under the 300 pound loading, from which the ratio of increase of the average stresses at mid span and over the supports may be computed.

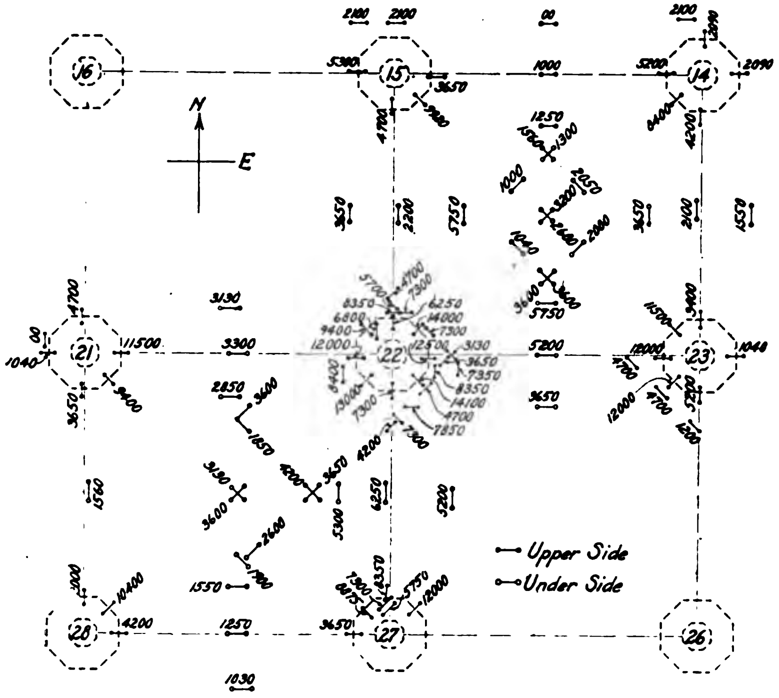

Fig. 142.

Steel Deformations times E_s, for 300 pound loading.

It appears by comparing the values in Figs. 141 and 142, that the rate of increase of the stresses at mid span was 2.1 times as rapid as the rate of increase in the loading between 300 and 900 pounds at mid span but only 1.45 times as rapid over the supports, indicating that the relative proportion of resisting moment is greatly decreased over the support with increase in the load and increased between the supports as would be expected from the large deformations at the edges of the caps in the concrete before noted under the greater load.

Under the load of 913 pounds, the yielding of the concrete at the edge of the cap on the under side of the slab caused cracks to open up on the upper side of the slab, as shown in the sketches in the bulletin which are reproduced in Fig. 143.

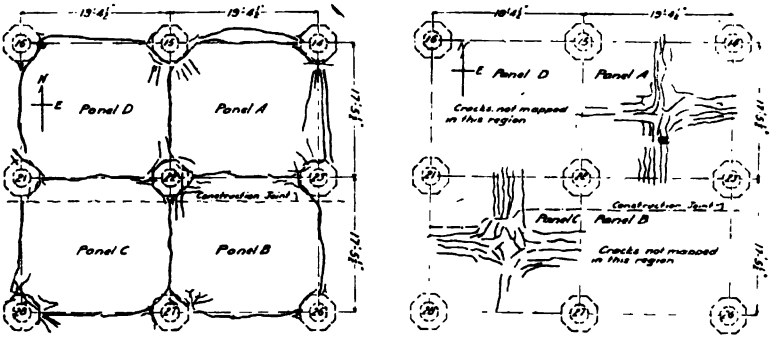

Fig. 143.

Figure Showing Cracks under the 913 lb. loading.

> "The cracks on the upper side at the load of 913 pounds per square inch opened to a width of .02 inch and .06 inch. Those on the lower side were not so wide. Upon the removal of the load the cracks closed, leaving the surfaces of the slab with the appearance which they had before the load was applied." (Bulletin p. 44.)

The comment on the cracks is that they may be expected to indicate the region of high tensile stresses in the reinforcing bars, a statement with which the authors partially agree, beacuse the cracks indicate localities of sharp bending and high fiber stress.

Column Stiffness. As the tested floor was an intermediate floor with rigid columns above and below, little bending would be expected to occur in them and an examination of the slab deformation about the columns bounding the loaded area taken in connection with the laps shown in Fig. 4 of page 12, of the Bulletin, indicates that the restraint at the outer edges of the panels was such that the angle of inclination at those edges was very small indeed so that the slabs might for practical purposes be considered as fully restrained.

Treating the edges of the loaded areas as fully restrained, we may investigate a panel wide strip East and West through the central part of the loaded portion and from the average stresses at mid span determine with considerable precision the moment of resistance along the column line by deducting the moment of resistance at mid span figured in accordance with the principles of Chapter VIII from the total applied bending moment and then apportioning the remainder between supports, and so determine the total resisting moment necessary for equilibrium at each support.

From these relative resistances we may determine the relation of the moment of inertia at mid span and over the supports and we may compare the result computed in this wise with that which would be figured in accordance with the principles in Chapter VIII. Then if these two different methods of investigation produce concordant results, we may proceed to separate approximately the fiber stresses from the axial stress in the measured steel deformations over the supports.

Under 300 pounds per square foot, the total applied bending moment is 1,280,000 inch pounds the long way of the panel.

Moment of arch action 103,000 in. lbs. or 8% for $E_c = 3{,}750{,}000$

$I_0 = 514$

$I_x = 250$ for column 22, while for columns 14, 26, 27 it is approximately 400 due to the laps of the slab belts.

The mean value of $I_x/I_0 = .65$

$C/L = .22$

$m_0 = {}^*.048 \times 98{,}000/2 \times 19.4 \times 12 =$ 545,000 lb. ins.

Less arch moment 103,000 " "

Applied moment in steel computed from loads = 442,000

Measured resisting moment

4.6 sq. in. $\times 3400 \times 2.5 \times 7.6 \times .87 = 260{,}000$ lb. ins.

$4.6 \times 5300 \times 7.8 \times .89 = 170{,}000$ " "

430,000 " "

In selecting the above average values it will be noted that the highest stress in both panels *A* and *C* about mid span did not occur at the center but at one side of the center due to the irregularity of the vertical position of the various belts in their emplacement on the column tops. Thus in panel *C* the maximum stress in a diagonal slab rod at mid span is 4200 at one side of the center and in panel *A* it is 3600 also at one side of the center and nearer column 22. Those particular diagonal belts are low in position over 22 and high in position over 28 and 14 respectively accounting for this shifting of the position of maximum stress. The splice laps were extremely irregular, massing metal between columns 14 and 15 to double the cross section; between columns 22 and 23 on direct belt lines, an equivalent average stress on the basis of a sectional area of four belts between columns is estimated as above and appears to agree reasonably with the applied moment computed from the loads.

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*Interpolation from tabular coefficients.

m_x from loads = 1,280,000 − 545,000 = 725,000 lb. ins.
Less arch moment 103,000 " "
Approximate moment of steel = 622,000 " "
Stress over Col. 22
622,000/(4.6×2×.7×2.5×.85×5.5) = 8450 average axial steel stress. Fiber stress $\frac{i+\frac{1}{2}d}{i}$ would increase the value approximately 1000 lbs. per sq. in., or to an average of 9450. This average value would in turn be increased for the lower rods having the smaller lever arm and decreased for the rods in the belts higher up and where the single belt is spliced the measured stress would be reduced in proportion to the increased section of metal.

A wide variation in stress is caused by the change in thickness of the concrete from 7.51″ to 8.52″ around the periphery of the cap taken in connection with the great thickness of the mat of steel. The top surface of the rods was found in the top layer to average .9″ below the top of the slab and the lowest layer 3.65″; therefore the distance from the center of the steel slab rods to the outer fiber of the concrete at the edge of the cap averaged from approximately 4″ minimum to 7.3″ maximum inducing a corresponding variation of from 25 to 30 percent above and below the mean axial resistance of the mat as a whole. The highly unsatisfactory character of the mat of such large rods in so thin a slab thus becomes apparent.

As the load was increased to 913 pounds per square foot the concrete tended to yield at and about the edge of the column capital. Such yielding would reduce the lever arm *jd* of the steel and cause the neutral plane to move toward the steel producing a material reduction in the value of the moment of inertia of the steel over the column and a corresponding decrease in the relative amount of moment to be borne at the column and an increase in the moment to be resisted at mid span. Thus in the diagonal belts in the Fig. 142 showing the relative stresses for 300 pound loading the highest stress in a diagonal direction is 4200 lbs. while under the 913 pound loading the highest stress recorded is 24,000 pounds, an increase of 5.6 times for an increase in loading of a little more than three times indicating a shifting of the apportionment of the moments because of the reduced moment of inertia by reason of over-strain of the material around the column support.

The diagram of deflections, Fig. 136 shows a marked yield point of the slab as a whole shortly after 600 pounds per square foot live load has been exceeded. The bulletin notes that there was a lack of recovery of 25 percent on removal of the load. This lack of recovery would represent in a large measure a permanent bending of the bars about the support with permanent compression of the concrete at the column edge so that no doubt approximately one third of the measured stress of Fig. 141 showing the fiber deformations multiplied by E_s for 913 pound loading represents bending of the steel brought about by the yielding of the concrete at the edge of the capital.

The average axial stress would of necessity be less than 25,000 pounds per square inch because of the reduction of the relative moments of inertia at the supports due to the excessive loading.

It thus appears that more than 50 percent of the measured fiber stress has no relation whatever to the axial stress.

Steel Stresses Caused by Shear. Under 300 pounds per square foot loading the bending stress induced by the coaction of the metal and concrete around the column cap is estimated to be about 400 pounds per square inch. As the load is increased to 900 pounds, since this fiber stress would increase in proportion to the square of the load, for the reasons before deduced, this fiber stress which would be in the neighborhood of 3600 to 4000 pounds per square inch would disappear as the load was removed.

Conclusion. It is difficult to verify the computed mode of operation by measurements beyond the yield point of the coacting materials experimented upon. For illustration, should we undertake to determine E_s from a piece of steel when deformations of the load approached the ultimate strength of the piece we would be dealing with an elongation of approximately 20 percent of the length in a specimen of structural grade steel and our value for E_s would be approximately 300,000 instead of 30,000,000 its true value. The error is not so marked when dealing with the coaction of metal and concrete as with steel alone but the method in the former case is equally erroneous.

Had the recovery at the various gage points been recorded in this experiment it would have added greatly to the value of the test.

The attempt in the Bulletin to conclude from this test that the operation of this particular slab is accounted for by beam theory is based on the following fundamental errors:

1. The assumption that the applied bending moment is equal to the statical moment by wholly disregarding the twisting moment which is a component of the statical moment.

2. An attempt to compare the computed resisting moment, with the applied moment on the naïve assumption that the tensile strength of the concrete accounts for results produced, by the operation of twisting resistance.

3. An equally fundamental error is also involved in the method of arriving at the applied moment. The presence of the lines of zero shear which determine the magnitude of the moment were not considered, while the moments required by the known position of these lines establishes a rule wholly different from the rule-of-thumb method proposed by the Joint Committee.

4. The sum total of the moments between mid span and supports accords with the law of moment magnitudes and the approtionment of them in detail depends on the relation of the moment of inertia and the moment areas, whereas the division attempted in the bulletin disregards this fundamental relation and assumes that the moment resisted over the support is twice that at mid span which is wholly unwarranted by the distribution of the steel found in the floor.

5. Such practical considerations as the effect of massing of steel are also disregarded in the Bulletin. For example, the statement is made in it that there is no apparent reason why the stress in the North and South belt should be but half that in the East and West belt when in fact there is twice as much steel present in it to resist the applied stress.

That a slab so poorly designed and inaccurately executed in fact carried four times its working load with the steel so arranged over the supports as to operate at half its normal efficiency as a mat and the depth of the concrete so reduced by the thickness of the mat of steel as to have but half its effective depth, speaks volumes for the toughness of such multiple way reinforced concrete slabs. To conclude as was done in the Bulletin, from such an extremely irrational design that the steel stresses were in any way typical of a normal design is wholly without warrant or justification.

12. Tests of a Concrete Floor in Abnormal Condition. *The Bell Street Warehouse, Seattle, Wash.* This building is of flat slab Mushroom type 80′ by 435′, five stories and basement. Four panels were loaded, two interior and two next to the wall. Panel sizes 20′9″ by 20″ interior panels and 20′9″ by 19′4″ for the wall panels. Interior columns 32″ octagons reinforced with sixteen 1-1/8″ round rods. Exterior wall columns 16″ deep, by 4′ the long way of the building, reinforced with. eight 1-1/8″ rounds and four 1″ round verticals Caps were plain, flared caps 24″ deep by 12″ spread. The slabs $10\frac{3}{4}$″ thick in the rough reinforced with seventeen 1/2″ slab bars in each direction.

The work was designed by A. O. Powell, Engineer, Seattle, but the inspection and supervision of the work was under the direction of the Port Commission. Standard specifications as to the handling of the concrete were supposed to be followed. The work was executed in winter time and although the concrete was initially heated it was thoroughly chilled by spouting it into position through long spouts and was somewhat frosted. The effect of this chilling was to cause the hardening to progress at an abnormally slow rate, such that at the end of 90 days the concrete was neither as rigid as it should have been normally, nor thoroughly dried out.

The test was conducted by Messrs. Fuller and More, assisted by employees of the Building Department.

Realizing the necessity of exact emplacement of the steel if results in keeping with theoretical computation were to be secured, the junior author requested that he be notified when the steel was placed in the test panels that he might independently make a careful survey of the emplacement. This request was overlooked and in place of compliance, a photograph Fig. 144, apparently indicating the good intention to show that this was done, was forwarded to him, but the final steel survey of the finished floor disclosed a material displacement of the exterior column head steel at the wall.

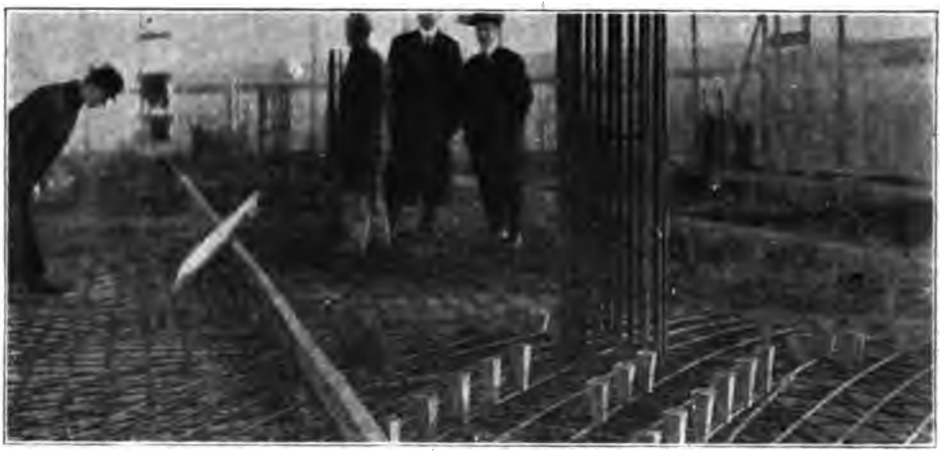

Fig. 144
Photo of the inspection that failed to secure correct vertical emplacement of the radial reinforcement.

The center of the slab rods in the direct belt (on which the gage line 350 was located) were required by the plan to be $8\frac{1}{4}''$ from the bottom of the slab; through error these were placed 5-7/8″ from the bottom of the slab. Accordingly, the moment of inertia was approximately 45% of what it should have been at this column head, had those conducting the test made a proper steel survey before allowing the concrete work to proceed. The range of elastic coaction of the metal and concrete determined by M/I was consequently less than one half of what it would have been had the observers been accurate inspectors.

Such a radical difference in the moment of inertia over the column makes a wide variation in the moment to be supported at mid span and over the columns, as heretofore shown, and furthermore, it reduces the yield point value of the coaction of the metal and concrete as will be discussed more in detail later.

The slow curing of the concrete tended to cause abnormal shrinkage stresses at the junction of the cap and column and some little permanent deformation of the concrete at the edge of the cap under initial loading. This loading would tend to relieve these concrete stresses. The extension of the upper fibers due to the sharp bending of the slab rods over the edge of the cap was incorrectly recorded as axial deformtion but would not vitiate the result were the recovery readings employed as the basis for determining the axial fiber stress.

Experimental work was very carefully done and the recovery readings noted and recorded in all cases. These recovery readings by showing closely the true axial stress in the bars give a clear idea of the change in deportment under load due to both the shrinkage stresses and the displacement of the steel. The great reduction in the moment of inertia over the wall columns induced sharp bending and slipping of the concrete and metal at that point, producing certain permanent deformations of the slab similar to those produced by the change in level of the supports of a continuous beam.

The lack of recovery at gage points 147, 148, 149 in the bottom of the slab between the wall column and the first interior column was practically 50% indicating that this percentage of the measured deformation represents permanent curvature due to the permanent deformation of the slab because of the reduction in the moment of inertia at the column due to the steel displacement noted.

Gage point 350 for example shows a deformation which multiplied by $E_s = 42,000$ pounds under applied load of 700 pounds per sq. foot, with a lack of recovery of approximately 17,000 pounds. At 306 directly over the edge of the cap it showed a deformation $\times E_s =$ approximately 22,000 pounds, and a lack of recovery of substantially 7,000 pounds, while 305 on the same rod just beyond the edge of the cap where it would be less influenced by the inelastic compression of the concrete showed only 13,000 pounds, with a lack of recovery of 5000. Gage points 316 and 318 show similar phenomena. Point 316 directly over the edge of the cap shows original deformation multiplied by E_s corresponding to 26,000 pounds with a lack of recovery of 9000 pounds, while 318 next to it and within the cap shows 21,000 pounds with a lack of recovery of 9000. Point 319 at one side of the edge of the cap shows the same phenomena.

The writer took exception to the results of this test on the ground that, in view of the low temperature when the concrete was cast and the failure to get it in place hot, the concrete had not become rigid and thoroly cured and that it was not for that reason a satisfactory test.

Second Test of the Bell Street Warehouse Discussed by Turner. The second test of the Bell Street Warehouse was made because exception was taken to the first test on the grounds stated. Because a more accurate measure of elastic axial stress may be obtained by repeated loading, the Department was advised by the writer, under date of August 12, 1916, as follows:

> "It seems to me that these (same panels) ought to be tested again so that a definite and indisputable answer could be given to the question as to whether the concrete was tested in an uncured condition as I maintain. If a test is desired in a new place it is all right, but the old slab should be tested to refute the wide publicity given to a premature test."

The request that the same panels be retested was made in order to show the indisputable improvement caused by further curing, and make a clear disclosure of the theoretical error in interpreting the readings of the first test from initial deformation instead of from recovery readings. But no attention was paid to this request. The readings under repeated loading would have been less than the former recovery reading on removal of the initial load because of the longer period of hardening.

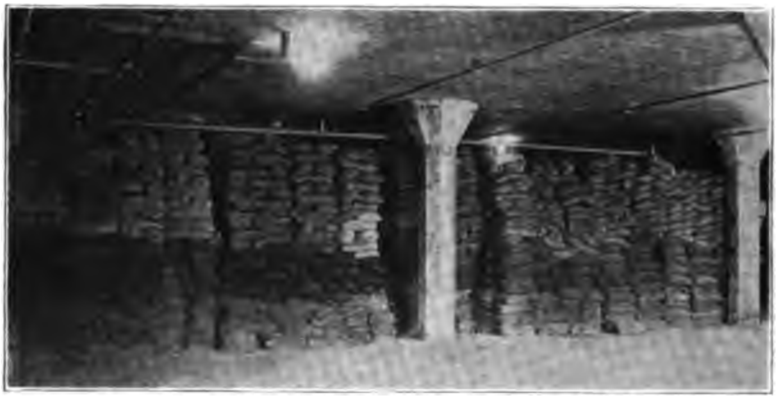

Fig. 145
Test Load Ore Concentrates in Sacks, 860 Pounds Per Square Foot Over Four Panels.

The Instrumental Work. The instrumental work was carefully done, little variation appearing in observations of the different observers. The readings accordingly represent the distances between the points of the instrument at the time the observations were made with accuracy. The interpretation of such readings or their correct translation into stress is dependent on the applied knowledge of the properties of the material on which the measurements are made.

Properties of the Steel. The reinforcing steel in the slab rods, as is stated by Messrs. More and Fuller, page 11 of their report of the first test, showed an ultimate strength of 90,000 pounds per square inch with a yield point value greater than 50,000. The steel accordingly is hard grade, which shows no set of sufficient magnitude to be measured by an extensometer under about 45,000 pounds per square inch. Therefore readings upon rods not stressed to this extent do not indicate permanent set. The small nick produced by drilling of the holes might affect the small slab rods so that the above limit is slightly reduced perhaps to as low a value as 40,000 pounds per square inch.

The distance between the points of the instrument is affected by the curvature of the rods as for instance where there are any kinks followed by straightening of the bar under stress, or by any bending of the bar with the bending of the floor slab, as well as by the temperature effects and the readjustment of initial shrinkage stresses in the concrete.

Altho observations on the recovery of the steel on removal of the load were made by both Mr. Fixen and the observers for the Department of Buildings, these important data do not appear in the published report.

The observations of both the initial deformation and recovery readings are tabulated in the accompanying tables I, II, III and IV. Column 2, gives the difference in reading on application of the load of 860 lbs. and the original zero reading. Column 3 represents the difference between the reading under a load of 860 lbs. per square foot and the reading when the load was entirely removed or the recovery reading. Table I shows the original readings on points 101 to 120 in the bottom of the slab excepting 105, 106, 115 and 116, which are given in Table III.

TABLE I.

Gage Line	A Reading at 860 lbs. load	Reading 860 lbs. removed	C Difference or Recovery	Percent Recovery	72.5A	72.5C
101	415	196	219	53%	31,200	16,500
102	345	178	167	48%	26,200	12,600
103	346	168	178	52%	25,500	13,300
104	348	179	169	48%	27,000	13,000
107	431	197	234	54%	32,500	17,600
108	448	213	235	52%	33,300	17,300
109	412	142	270	66%	30,700	20,200
110	403	178	225	56%	29,800	16,700
111	283	139	144	51%	21,700	11,100
112	315	105	210	67%	23,900	16,000
113	351	128	223	64%	25,400	16,300
114	310	147	163	53%	24,100	12,800
117	241	99	142	59%	15,500	9,200
119	351	132	219	62%	21,700	13,500
120	315	127	188	60%	20,000	12,000

A and C are multiplied by the constant of the instrument, 72.5 to obtain the product of deformation times modulus = stress. 56.3% average recovery

TABLE II.

Gage Line	A Reading at 860 lbs. load	Reading 860 lbs. removed	C Difference or Recovery	Percent Recovery	72.5A	72.5C
306	379	83	296	78%	27,500	21,500
307	648	138	510	79%	46,500	36,700
308	531	161	370	70%	37,200	26,00
309	801	276	525	65%	58,200	37,800
310	667	168	499	75%	48,700	36,600
311	522	181	341	69%	37,800	24,600
312	552	177	375	68%	40,700	27,700
313	707	....	....	....	52,700	......
314	387	54	333	86%	28,200	24,200
315	533	214	319	60%	37,900	22,800
316	585	150	435	74%	42,900	31,800
317	409	85	324	79%	30,200	23.800

Multiply by 72.5 for product of deformation times modulus. 72.6% average recovery.
Above rods are average of those around top of slab at interior column No. 72.

TABLE III.

Gage Line	A Reading at 860 lbs. load	Reading 860 lbs. removed	C Difference or Recovery	Percent Recovery	72.5A	72.5C
105	689	248	441	64%	50,800	32,600
106	709	286	423	60%	52,600	31,600
115	340	129	211	62%	24,400	15,100
116	306	140	166	54%	21,400	11,500
				Ave. 60%		
301	447	65	382	86%	33,500	28,800
304	668	214	454	68%	48,500	33,000
305	811	323	488	60%	60,200	36,100
				Ave. 71%		

Multiply by 72.5 for product of deformation times modulus.

TABLE IV.

Gage Line	A Reading at 860 lbs. load	Reading 860 lbs. removed	C Difference or Recovery	Percent Recovery	72.5A	72.5C
201	252	105	147	58%	1,700	985
202	314	145	169	54%	2,100	1,130
203	485	182	303	62%	3,100	1,920
205	338	108	230	68%	2,100	1,430
				Ave. 60.5%		

The percentage of recovery in these gage lines in the bottom is found to vary from 48% to 67% or an average recovery of 56.3% leaving a residual of 43.7 percent. Interpreting the stresses on the basis of 72.5A all of these stresses are low relatively to the yield point of the steel. The highest on gage line 108 is only 33,000 lbs. with a recovery of 52% while gage point 102 shows 48% recovery with a nominal stress, computed in this manner of, 26,000 lbs. Steel is elastic within the yield point while the concrete on the other hand is by no means elastic until it has been subjected to repeated loading.

The proper interpretation of extensometer measurements to indicate axial steel stress should be based on measurements of recovery after removal of the load or measurements made on repeated loading to bring out the precise state of facts as to the steel stress since this stress is produced by the coaction of the metal and concrete. If the indication of lack of recovery is due to permanent set or to an actual deformation of the steel instead of being partly produced by a change in distance between the points of the instrument thru curvature of the bar or axial change in direction of the small reinforcing bars then the percentage of recovery should decrease with a large increase of observed deformation.

The readings both initial and with respect to recovery made in the upper slab rods over the top of interior column 72 are presented in Table II.

The initial deformation times E_s varies from 27,500 to 58,000 so that they are much higher than those of Table I but the average recovery instead of being less than 57% is nearly 30% greater, or 72.6% so that it appears that the higher the steel stress in the slab tested the more perfect the recovery toward the initial reading before loading, a result wholly at variance with the deportment of the steel bar by and of itself and one which would at once cast doubt on the assumption that the measurement represents pure deformation instead of curvature and deformation combined.

In Tables I, II, II and IV, Columns 2, 3, 4, and 5 are based on Mr. Fixen's readings. Columns 6 give the values reported by Mr. Hooker under the 860 lb. load. Column 7 is obtained by multiplying values in Column 6 by the corresponding percentage as shown in Column 5 or by multiplying C by 72.5.

The observations on gage points 115 and 116 are closely duplicated by 119 and 111 both as to intensity of stress and percentage of recovery. Also gage lines 105 and 106 in which the bars are much more heavily strained present a percentage of recovery materially higher than the average under Table I in which the observed deformations were small. It therefore appears that the more highly the steel was stressed the more complete the recovery of the steel on removal of the load under the conditions of this test as shown by these extensometer measurements, and our attention is therefore directed to further investigation of this phenomenon which indicates material errors in the interpretation of readings in the published report.

It is a reasonable and accepted conclusion that parallel rods of the same size and placed in the same manner and at the same distance from the neutral plane are stressed alike and any experimental determination incompatible with this rational conclusion must have a most substantial foundation before it can be accepted by thoughtful members of the profession.

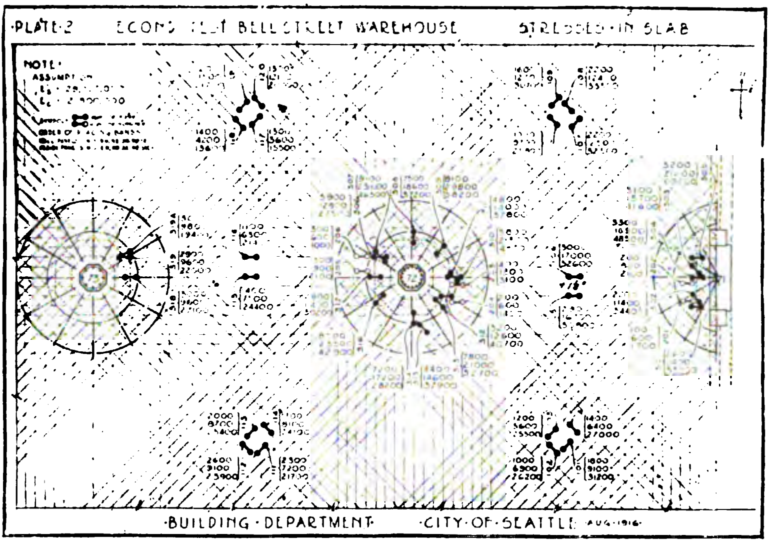

Fig. 146. Plate 2 of Report.

We will now compare readings on similarly located gage points 301, 304 and 305 over the capital of wall column 71. (See Fig. 146 Plate 2 of the Report.) These readings were taken as indicating pure stress under the final load of 860 lbs. per sq. ft. for gage line 301 of 33,500 lbs., for gage line 304 of 48,500 lbs., and for gage line 305 of 60,200 lbs., per sq. in. But here is a variation of approximately 100 percent from the common assumption of design so that if this interpretation of measurement of stress is correct, and if it has any foundation in fact, the ordinary rules of design should be modified at once because these three 1/2 inch rods are parallel rods similarly located and there is no conceivable reason why the stress actually sustained by them should vary more than 5 to 10 percent from the mean value.

Considering now the recovery of these bars after the removal of the load and reducing the measured change of deformation under recovery to stress we find that 301 Table III recovered 86 percent, 304 recovered 68 percent and 305 recovered 60 percent, so that the extreme of the resulting stresses differ from the mean by but 10 percent when the errors due to change of curvature, straightening out kinks or little bends which were undoubtedly made in the bars in the fabrication are eliminated. It is therefore apparent that consideration of the change of deformation under recovery is a more rational interpretation than one which may involve the gross errors above noted.

There is yet another line of reasoning by which the same conclusion may be arrived at. This is furnished by the well known law governing moment magnitudes which shows that the numerical sum of half the moments over the supports plus the moment at mid span is a constant. Consequently if we have the readings on each of these bars at mid span and over both supports we should be able to get a line on the probability of the variation of 100 percent in the stresses over the wall column, but apparently when we consider gage points 105, 106, 310 and 312, which is the nearest that we can come to comparing these readings, we have a refutation of this law in the measurements thus interpreted, hence a refutation of the validity of the method of interpretation adopted in the report.

Properties of Concrete Coaction with the Steel. Concrete unlike steel is far from being perfectly elastic. Shrinkage stresses are developed in curing and when a floor is first loaded some permanent deformation occurs following the relief of these stresses, and a readjustment between the relations of the steel and concrete in their coaction together follows. These relations are so complex that little effort has been made to investigate them or their cause by the research department of our universities. The intensity and amount of the shrinkage stresses depend upon the plasticity of the mix of concrete, the contour of the forms, the rapidity of pouring and hardening and how soon relatively to the progress of hardening the superposed loads are brought upon partially cured columns during construction.

The Bell Street Warehouse is exceptional in respect to certain conditions of casting and curing. The concrete was spouted at a temperature of about 32 degrees F. To facilitate this method of deposition it was mixed with more water than is desirable and it was deposited rapidly by this method. Because of the low temperature and the long spout, the concrete which was mixed with warm water was cold by the time it was in place and appears to have been touched by frost. It was exceedingly slow in curing and undoubtedly received some set by the rapidly superposed load of the upper stories causing unusual shrinkage stresses about the cap by the compression and flow of the partly cured concrete. This would accentuate the condition of internal stress from shrinkage at the edge of the column cap where these stresses are of necessity greatest.

The concrete at the bottom of the slab and next to the periphery of the column capital was in a state of tension due to this shrinkage and inevitably took a permanent set of considerable magnitude under the first loading. This loading relieved the shrinkage stress and produced some permanent deflection at mid span. The set produced under either the working load, or greater loads, caused a change in the curvature of the steel above the edge of each capital, and a relatively large apparent deformation due to the bending of the rods, which does not represent axial stress in the steel except in part, and this should be apportioned by noting the apparent recovery of the steel on removal of the load. The curvature of the slab and steel as well, occurring at this point is evidently much sharper than the total residual deflection of the slab would indicate.

The question naturally arises, how can the steel accomodate itself to the change of form of the slab without being strained beyond its yield point? It may be noted that the steel is placed in a sinusoidal curve which permits readjustment with the concrete without elongating the steel as would of necessity result were the steel disposed in a catenary curve. That the steel under such moderate loads as those placed upon the Bell Street Warehouse is not in any wise overstrained is shown by the amount of deflection. Slabs of this span and thickness have been tested without injury to as great a deflection as one and one half inches where the columns were not particularly rigid, and with a recovery of 85 percent.

Investigation of Errors. The concrete and steel both work together. When the concrete takes a permanent set the form or contour of the slab is changed and the bars bend. Bending stresses are readjusted in small bars under tension so that they have little effect upon the ultimate axial resistance of the bar, providing, of course, that the bending is not too sharp. That this is the fact is generally unconsciously recognized by all engaged in practical construction since bars are usually bent cold through a large angle of curvature and yet they are not thought inferior in resisting tension.

Accordingly, small steel fiber stresses due to change of curvature should not be considered in rating the safety of the structure nor should they be considered as in any wise representing axial stress in the bar, which it is the object of the experiments to determine.

Again, it is known that any change in curvature of the bar has a marked effect on the extensometer readings other than and different from the fiber stress plus the axial stress. Preparation for extensometer readings is made by drilling into the rod with a small drill of about 1/20th in. diameter and burring the edge of the hole so that the line of bearing on the conical points of the instrument are approximately or a little more than 1/16th in. in diameter. Any bending of the bar, whether from the straightening of some kink or curvature of the slab causes shifting of the axis of the fixed point relative to the axis of the hole. Knowing the angle of curvature and taper of the conical point it is a simple matter of solution of a plain triangle to determine the amount of this shifting for the fixed point of the instrument. This is not true for the movable point or leg since the angle of curvature, by changing the distance between the points on the outer fiber of the bar changes the angle of the intersection of the axis of the leg with the axis of the hole and increases this shifting of the point relatively to the axis of the hole much more rapidly than the curvature of the rod at the fixed point.

Computation on this basis will show a large error for a change in the direction of the tangent of the bar of a single degree. This error increases rapdily with the increase of the angle so that large differences in readings will be found wherever these small slab rods have been slightly kinked in handling and placing and change in curvature has occurred under test conditions. Evidently when the concrete receives a permanent deformation from the elimination of shrinkage stresses through yielding of the concrete or by excessive bending, the bent bars do not come back to their original curvature as the load is removed, although no axial set or permanent stretching has taken place in the steel. While the amount of these corrections may be computed if we assume the amount or degree of change in curvature, there has been developed, to date no method of measuring this change in curvature accurately nor has such a method been developed from a consideration of elastic and inelastic deflections.

Whatever the apparent deformations in the steel may be under a residual deflection due to inelastic deformation of the concrete after removal of any load that has not stressed the steel beyond 16,000 or 18,000 lbs. per square inch. it is certain that the steel has received no permanent axial set at such a low stress as this and any residual steel deformations must be due to bending which has occurred during loading and not during recovery. These deformations do not indicate residual axial steel stresses. Consequently, the most accurate measure available, in this case, of the stress that existed in the steel before the load was removed is obtained by multiplying the modulus

E_s by the amount of the recovery after removal of the load. A still more accurate method, however, would be to repeat the loading of the same panels and observe the deformations produced thereby. A procedure which would avoid the influence of inelastic deformation of concrete and the curvature of the steel.

In Chapter XII, in our treatment of unbalanced load, we have shown how the stiffness of the column enters into the operation of the slab by determining the angle v of the inclination at the support. In the tests of the Northwestern Glass Building, Franks Building, and Larkin Building, the rigidity of the columns rendered this angle small indeed.

In the Bell Street Warehouse, the small dimension of the wall column normal to the side of the building does not afford the customary degree of restraint for such columns. This matter was discussed at some length with Capt. Powell, who desired a relatively thin but wide column in order to economize space next to the wall. The intent of the designer was to support a bridge on exterior brackets which would stiffen the wall column sidewise, while on the water front the necessary runway would prevent excess loading.

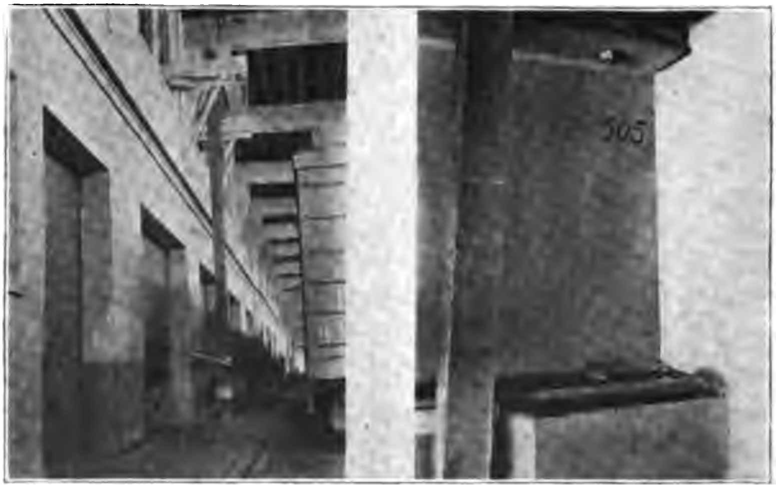

Fig. 147

Photo A, Showing jack screw and strut used for regulating the angle $\underline{v}$ at the wall column.

Photo B, Showing Bracket supporting bridge girder

The bracket is shown in the accompanying cut of Photo B while Photo A shows the effort of Engineer Hooker to regulate artificially the magnitude of the angle of restraint by jacking up the beam resting on the over-hanging bracket. It is difficult to suppose that this was done under the misapprehension that the building in use would require jack screws and a post under each of these various beams shown in Photo A, to remove the negative bending of the bridge floor in order to secure the safety of the building when this negative bending enchances and increases that safety.

The omission of any statement of fact regarding this matter in the published report is perhaps inadvertent.

The Displacement of Steel in the wall panel at mid span in the bottom of the slab was unusually great. The center of the rods at gage point 105 and 106 were 1-3/4" and 2-1/8" above the bottom of the slab instead of 3/4" thereby reducing, if these are average heights, the moment on inertia at mid span to 70% of that of a structure built as required by the plans. Over the wall columns the average vertical displacement was 1.44", reducing the moment of inertia to two-thirds of that which the vertical position of the steel required by the plans would give. While the ultimate strength of the floor was reduced perhaps not more than 12 to 15 percent, the test capacity was, by such displacement reduced at least 35 percent for the elastic coaction of the metal and concrete.

The apparent assumption in the report that the vertical displacement of the steel reduced the test capacity in proportion to the reduction of the effective arm of the steel before checking of concrete occurs is as has been shown wholly erroneous and is apparent at once from the fact that were the steel placed at mid depth or in the neutral surface it would be without effect in strengthening the slab until after the concrete had cracked almost in to the neutral surface in bending.

The data of this test furnishes the basis from which certain important deductions can be made.

First, true axial stresses in reinforcing elements can be accurately determined only from the recovery readings and not from the initial deformations where there has been excessive shrinkage stress developed in curing or where the bending under test has been sharp.

Second, the magitude of the common error in interpreting extensometer readings is brought out strongly in this test as the recovery readings were carefully recorded. No one would think of converting deformations beyond the yield point of a piece of steel into elastic stress by multiplying these deformations by the modulus E_s of the steel, yet in treating reinforced concrete structures it has been common to attempt to convert deformations which are partially inelastic by reason of the imperfect coaction of the metal and concrete from shrinkage and sharp bending into elastic stress by the very process that would be considered absurd in treating the steel element, alone. Thus the elastic stress in a bar of steel stressed beyond the yield point may be computed with approximate precision if the unit recovery is measured and this recovery multiplied by E_s and the same is true of the combination of metal and concrete.

Third, the criticism based upon checks in the concrete is in fact a criticism of the lack of thoroughness of the work of the investigators as inspectors in the first test in which the emplacement of the steel was inspected by them, and in the second test, the responsibility lies, not with the designer, but with the inspectors on the work. Where there is no inspector on a job, a reliable contractor places this work in charge of an ordinary carpenter who makes it his business to measure the vertical position of the steel and approves it before the pouring commences. On the other hand, where a force of inspectors is employed by an owner, the contractor considers himself relieved of such responsibility and pays attention only to the rapid execution of the work. Indifferent inspection by the owner is therefore, as a rule, worse than none at all.

Criticism of a design should in fairness be limited to features in which the design has been executed precisely according to the details thereof. Reinforced concrete details have rarely been prepared with greater completeness and precision in the details of emplacement of metal than were the plans of this building. Vertical displacement of steel from 1-1/2″ to 2-1/2″ in a slab as thin as 10-1/2″ does not come within the range of ordinary care. A resulting moment of inertia, which determines a sharpness of bending, less than half that which would result from following the design has a marked effect in causing cracks under loads which do not approach the ultimate strength but are sufficient to cause permanent change in the contour of the slab, relieving parts which are overstrained and permitting the resistance of other parts to come into action, by shifting the resistance to more rigid portions to an extent possible in reinforced concrete as it is in no other kind of structure.

Finally, in considering the stiffness of the columns in furnishing rigid restraint it may be observed that this rigidity decreases with the cube of the length of the column and varies as its moment of inertia. If the column merely rests at its foot it furnishes but half the restraint present when thoroughly anchored in the footing below. A column in a continuous line with sufficient load brought upon it from upper stories is enormously stiffer than the column of a deck slab in which the steel is forced to supply the tensile resistance under flexure.

With the counter moment of the bridge the wall column of the panels of the second test would be comparable in stiffness to an average wall panel, but with this restraint removed by energetic jacking the wall column did not present such rigidity.

Under the emergency of the recent war conditions it is said that these floors were loaded to more than their full working load over large areas and that under such loads, as should be expected from the foregoing analysis no indication of over-strain was observed, and unquestionably the axial stresses in the reinforcing elements did not exceed by more than 15 percent those figured upon notwithstanding inexcusable displacements of the metal which limits the overload capacity, without checking the concrete, to a much lower amount than it would have been had the steel been properly placed.

13. Summary. Uniformly reinforced concrete beams loaded at the one-third points store some 60 percent or more of the total energy of flexure in the steel reinforcement alone, and only about half as much in longitudinal compression in the concrete.

Continuous uniformly loaded column supported flat slab floors with mat reinforcement, store only about 12.5 percent of the total energy of flexure in the slab steel. Since this is only about one-fifth of the amount stored in beams, it evidently is impossible to compute slabs by beam theory.

The accuracy or inaccuracy of the elastic theory of reinforced concrete is not a matter of opinion but one capable of demonstration by the elastic behavior of the floors in accord with that theory and by the agreement of the measured stresses with those computed by theory when all elements entering into the phenomena are given due consideration. Experimental measurements will diverge from computed values unless proper allowance is made for subsidiary as well as predominant phenomena of flexure. As the elastic theory is based on the assumption that the direct tensile strength of concrete plays but a negligible part in resistance to transverse bending, measured steel stresses which are above 4000 to 5000 pounds per square inch indicate the accuracy of this assumption when due weight and consideration are given to the subsidiary phenomena presented by every test.

To reduce measured deformation to stress the recovery determination should be multiplied by the modulus of elasticity of steel for the same reason that it is necessary to do this very thing in the case of steel if we are to determine the elastic stress where the strain has exceeded the yield point value of the material. Because of induced shrinkage stresses the coaction of the concrete and metal is not elastic except for a limited angular deformation when the loads are first applied, and the measurement of the recovery on removal of the load is the one to be given the greatest weight in a scientific test.

Unless properly interpreted, extensometer measurements not only fail to clarify but add to present-day confusion and mystification regarding the mode of operation of concrete slab floors.

The calibration of an extensometer for curvature of the bar so that curvature and fiber stress may be properly apportioned is exceedingly difficult mathematically, but comparatively simple experimentally. Fix two points of the rod on a plank two to three feet apart dependent on the size of the rod and arrange a micrometer screw operating normal to the center of the line joining the two points. Place a square bar spanning the supporting points and apply the central screw. Clamp the bar in its curved position, compute the fiber stress and compare the difference of extensometer readings between the condition of the bar straight and the bar curved, and the correction for curvature for that degree of bending will be the difference determined.

This correction will vary widely for different sized bars and for different kinds of instruments with different lengths of leg, all of which requires investigation for dependable results in any test that is relatively severe or where the member is one in which relief of shrinkage strains may occur under initial load.

In the column supported flat plate scientifically designed, because the shearing resistances play a far more important part than in the operation of beam structures, the deflection curves approach more nearly a straight line, again confirming the difference in mode of operation of such a floor plate from that of ordinary beam action.

CHAPTER XIV

FOUNDATIONS

1. Bearing Value of Soil. In a building the floor loads are carried to the columns, or to the walls in case bearing walls are used, and the weight is concentrated on small areas of ground at the footings of the columns and walls. Evidently if we are to avoid settlement the weight must be distributed over a sufficient area. The following are suggestions for safe loading for foundations where the material can be clearly defined.

Ordinary ledge rock, such as good shale, limestone and the like, twenty to thirty tons per square foot. Granite, trap where the ledge is not shattered, fifty to seventy tons per square foot. Hard pan, seven tons per square foot. Gravel, five tons. Clean, coarse sand, four tons. Fine sand, with a little clay, three to three and one-half tons. Hard clay, three tons. Clay, such as is to be found in Regina, which is rather soft, not over one and one-half tons. Blue clay of Winnipeg, two to two and one-half tons.

In each case, however, it is well for the engineer to look into the conditions carefully unless thoroly conversant with the locality. His judgment as to the bearing power of the soil should be checked, if doing business in a strange city, by a careful examination of the buildings resting on similar foundation and general inquiry among experienced members of his profession. This caution may prove of value to the engineer doing business over an extended area, particularly if he is acting in a consulting capacity for a contracting firm assuming responsibility for the design.

In cases where there is filled ground, marsh, quick-sand and the like it is frequently necessary to use a pile foundation or distribute the weight over the entire area. In general in reinforced concrete construction, it is economical to use a comparatively thin footing and thoroly reinforce it and make the concrete a rich mix.

2. Column Footings and Method of Figuring. Fig. 149 shows a form of footing which seems somewhat objectionable for the reason that as usually placed the concrete is worked rather dry in order to make the slope without top forms, and the contractor does not ordinarily get it in place so that it can be depended upon with certainty.

A preferable form is shown in Fig. 149, in which the footing is made in two layers. The bottom layer is cast with the rods at the bottom, then the column assembled and the top layer cast with the column. Its advantage from a practical standpoint is first, that the upper layer of rods assists the footing in resisting the shearing strain of the column which tends to punch thru the footing and secondly it also assists in distributing the load out over the lower plate. The

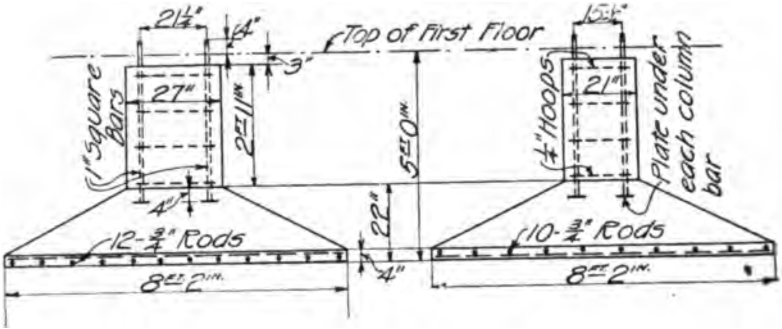

Fig. 148

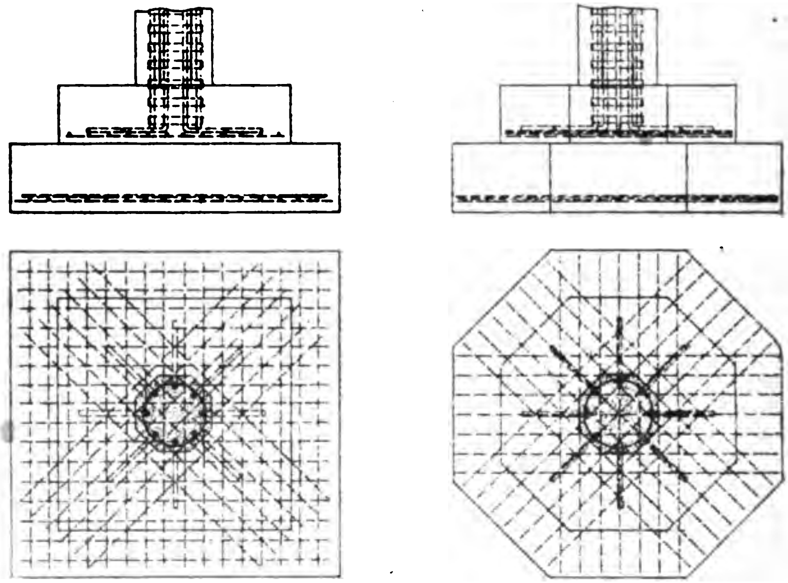

Fig. 149

bottom plate being cast with a wet mix we can depend with certainty on the bond between the steel and the concrete by the shrinkage of the wet mixture as we could not in the footing first mentioned.

In computing the lower plate, the cross sectional area of the steel should be such as to provide for a bending moment of the upward pressure on the under side of the plate distributed uniformly over it with the column as the point of support.

The arm of the total pressure either side of the center may be taken roughly as five-eighths of the half diameter of the footing, hence the actual bending moment is $5/16\ Wb$.* The metal acting as a flat plate would be as we have shown for the square panel twice as efficient as single way reinforcement and as the sectional area of each bar comes into play on each side of the center of the footing we can use four times the area of all rods crossing the footing times .85 of the total thickness of the two layers times the working stress on the steel as the resisting moment.

The amount of steel placed in the top layer is more a matter of practice than that of exact computation. The rods are generally made the same size as those used in the lower layer and no time is wasted in computing the stresses thereon. It is preferable to take long rods and bend them hair-pin style for the footings rather than to use short rods.

There is a further advantage in ordering the rods this way, that in case the steel for the footings is delayed, stock steel can be used while the steel ordered for footings may be used elsewhere in the building later.

3. **Pile Foundations.** Where piles if used will be continually wet and there is no possibility of changing conditions from that to alternate drying out and wetting, there is no type of reinforced concrete pile that can compete with timber piles. However, where the piles are liable to be above the permanent water line or it becomes necessary to excavate thereto, then and there concrete piles become an economic method of building up the foundation.

Practical concrete piles may be divided into two classes, one, those which are made up first and driven afterwards, and second, those in which a core or form is used and the hole filled in with concrete reinforced or otherwise.

4. **Driven Piles.** Hennebique was one of the first to use reinforced concrete piles that were made up separately and then driven.

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*In this notation *b* is the breadth or diameter of footing.

In design they were similar to his columns which have been previously illustrated, with longitudinal reinforcement coupled with either of several arrangements of lateral ties.

Considère in his pile design, shown in the accompanying Fig. 150, commonly made use of spirals for lateral reinforcement in place of Hennebique's ties. His form of piles is used to a large extent on the continent and is similar as regards the point at its lower extremity to Hennebique's.

In driving, it is customary to use a water jet for loosening up the sand or earth at the bottom and rap or jar the pile into place with a hammer. A steam hammer should be used in driving concrete piles rather than a drop hammer, since with the heavy ram and short stroke of the steam hammer there is less shock.

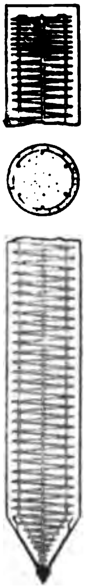


Fig. 150 Showing Reinforcement of the Considère pile.

5. Piles Cast in Place. Coming under the second class are the Raymond and Simplex piles.

The Raymond piles are made tapering to secure greater resistance against settling. The manner of constructing Raymond concrete piles is as follows: A steel pile-core the size and shape of pile desired is encased in a thin, closely fitting shell. The core and shell are then driven into the ground by means of a pile-driver in the usual manner. By a simple and ingenious device the core is collapsed or shrunken slightly, so that it loses contact with the shell, and is easily withdrawn, leaving in the ground a clean, perfectly formed hollow tube of the size and depth required, which has only to be filled in with the best Portland cement concrete to complete the pile.

Fig. 151, shows the Raymond pile expanded ready to be driven.

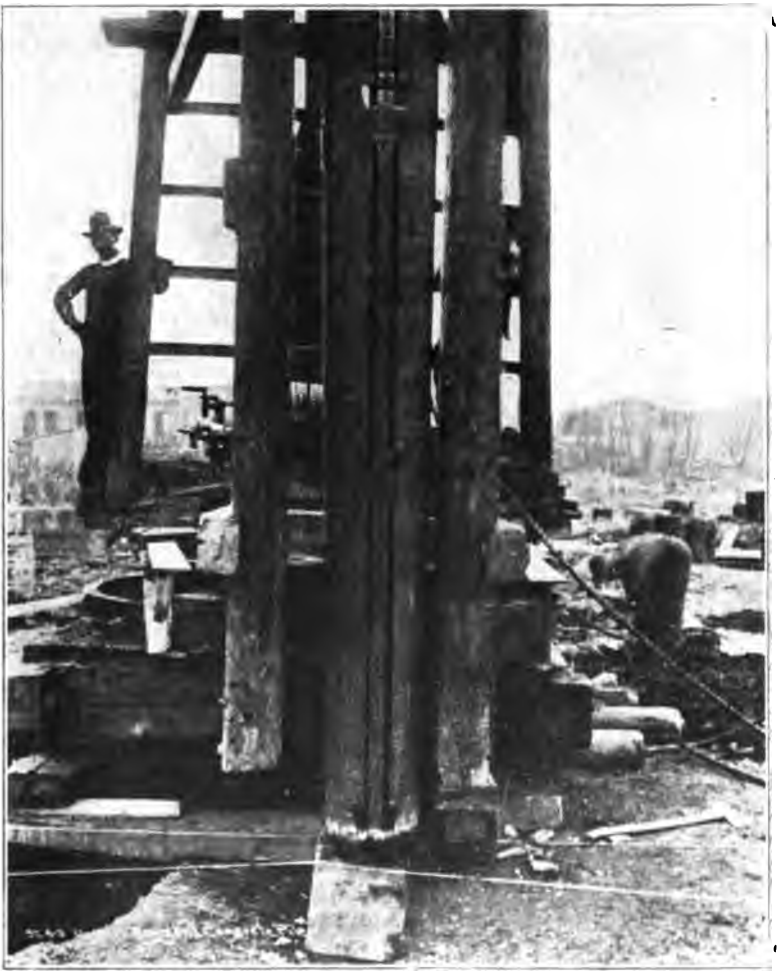


Fig. 151. Raymond Pile Core. This core is used to drive to rock and tapers from 20 to 13 inches. The cut shows core full size as it is driven.

Fig. 152, shows the core in the leaders with the shell on the right.

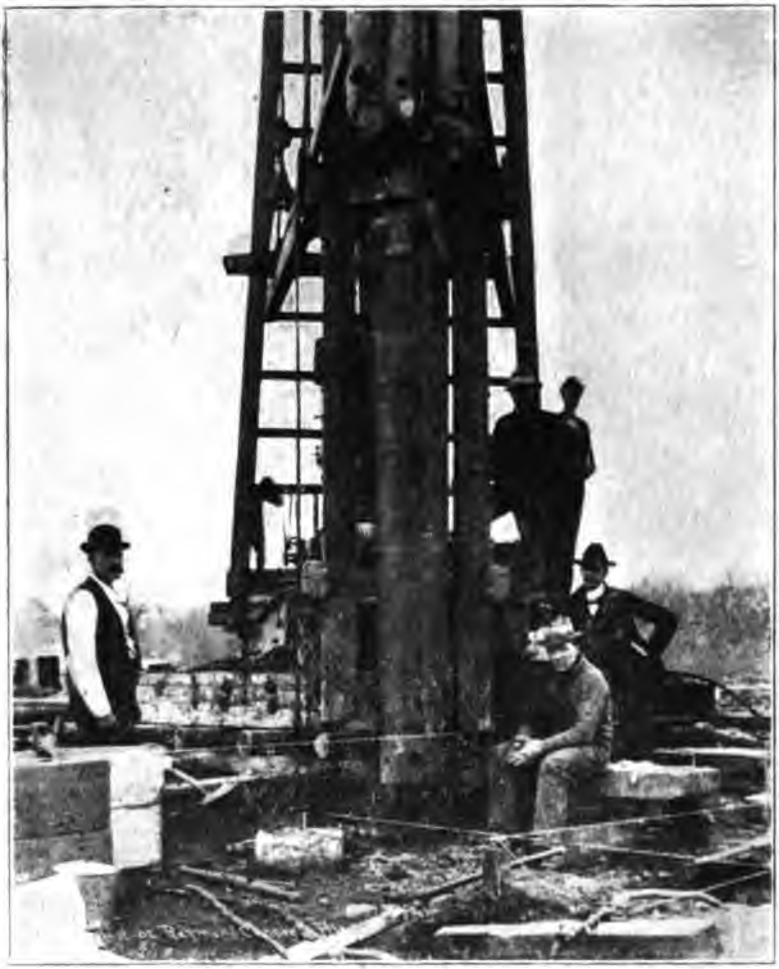


Fig. 152. Pile Core Collapsed or Shrunken, and partly withdrawn from the shell.

The core is expanded when driven and collapsed to be withdrawn from the shell.

The relative merits of the various kinds of piles would apparently depend on outside conditions. A pile like the Raymond pile should be most suitable for a clay soil where the consistency or cohesiveness of the clay is such that the pile core can be driven and the shell omitted.

Raymond piles are made of various lengths, tapering generally from 20″ at the top to 6″ at the bottom, making a symmetrical cone affording by friction material resistance to soil penetration.

For the Simplex pile a casing is first driven, and then as the casing is pulled up, concrete is deposited and rammed in place, forcing it out to a somewhat greater diameter than the shell that has been driven.

Figs. 153 and 154, show the make-up of the Simplex pile. With the steel casing is driven a point, either of steel or concrete, and afterwards the shell is gradually withdrawn and the hole filled with concrete as the shell is pulled up.

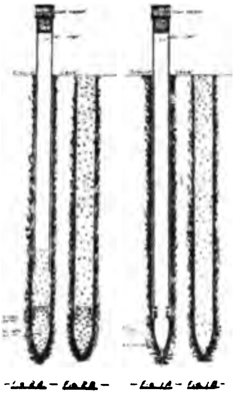


Fig. 153 Fig. 154
Simplex Pile.

For use in earth that is reasonably firm in its texture and free from water, the preparatory removable pile, (see Fig. 154) is used. This pile form consists of a length of extra heavy wrought iron pipe, fitted with a suitable driving head of oak, and a conical steel point of a somewhat larger diameter than the pipe, and fitted with an automatic air valve. This preparatory tube is driven into the ground to the required depth, and then withdrawn without difficulty, and the hole so produced is filled with well rammed concrete. This form of pile can be constructed of any desired length, as the preparatory tube can be driven and removed with but a fraction of the force required in the planting or removal of the ordinary pile. It can also be driven thru ground of a density quite impenetrable by any wooden pile and to almost any desired depth, as there is comparatively small increase in frictional resistance, as the depth increases, either in driving or withdrawing the tube.

The ramming process forces the larger pieces of the aggregate into the sides of the hole, materially adding to the frictional hold of the pile on all parts of its surface.

Where the earth is soft, marshy, or where quicksand or water is encountered, a detachable "point" of concrete, (see Fig. 155) is substituted for the fixed one of steel. This concrete point is driven to the required depth, and as the pipe is being lifted off, concrete is gradually filled in and rammed home thru the pipe, care being taken that a head of the concrete be maintained inside of the pipe while it is being thus gradually withdrawn. By this system all water is displaced and the possibility of the sides of the aperture closing in is entirely removed.

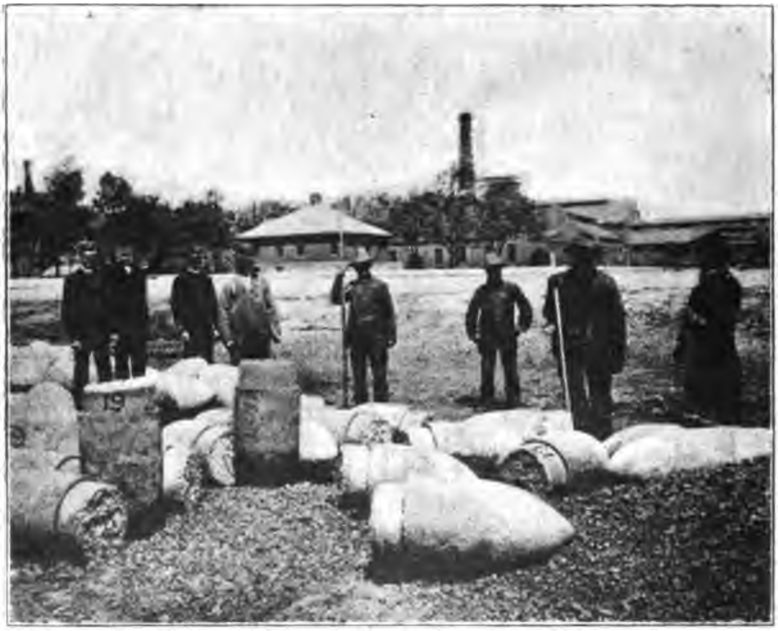


Fig. 155. Concrete Points for Simplex Piles.

Corrugated piles have been patented by Frank B. Gilbreth. One of the claims advanced in their favor is that the corrugations assist in jetting the piles into place.

Fig. 156 shows a driver handling one of the corrugated piles.

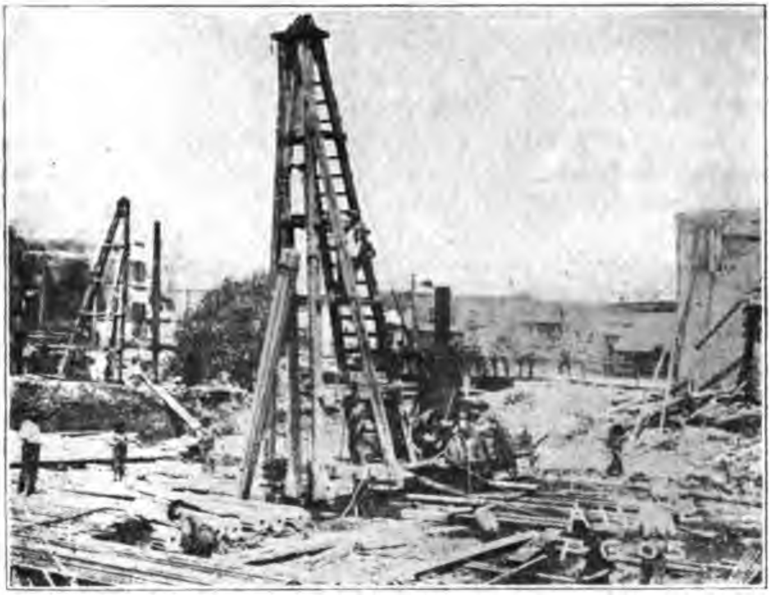


Fig. 156. Driving Corrugated Concrete Piles.

Fig 157 shows the cushion cap used in driving the pile.

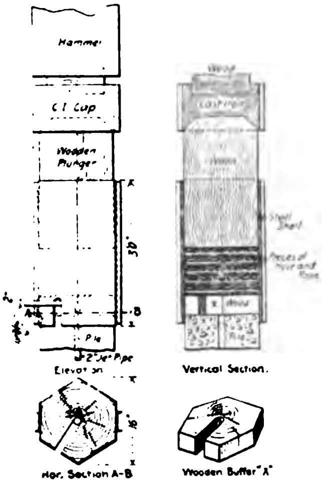


Fig. 157. Cushion Cap used in driving the corrugated concrete pile.

Concrete piles have this advantage over wood piles; they do not decay, are not subject to destruction by insects and furnish a durable foundation regardless of soil conditions; they can be used in dry filled ground as well as in wet soil, which may dry out and cause wood piles to decay.

A pile foundation generally tends to prevent settlement by the packing and settling of the soil in the vicinity of the pile, frequently rendering the bearing value of the soil around the piles materially greater. See Fig. 158.

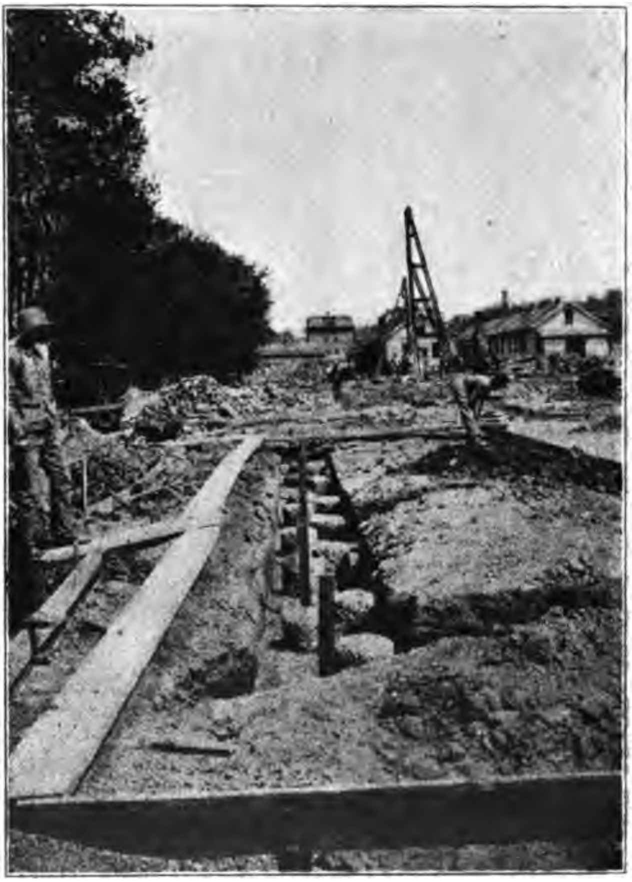
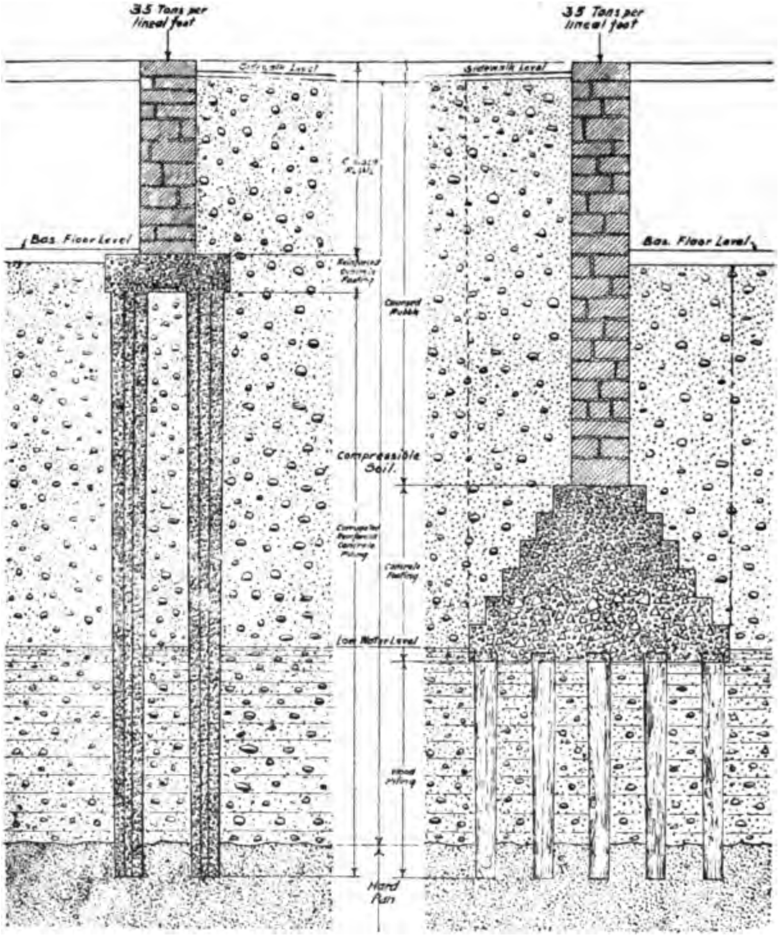


Fig. 158. A Trench Filled with Concrete Piles.

Fig. 159, gives a fair idea as to the advantage of a reinforced concrete pile foundation over the wood pile which must be cut off in the vicinity of the low water.



Reinforced Concrete Piles. Wood Piles.

Fig. 159. Showing the advantage of concrete over wood pile foundations.

6. Safe Bearing Loads for Piles. Two cases are to be distinguished; that of piles, the lower end of which rests upon a hard strata and that of the ordinary pile which is supported largely by skin friction of the material into which it is driven. The capacity of the former is determined by the strength of the pile as a column thru the upper or soft strata while the bearing power of the latter is some function of the penetration under a given drop of a ram of a given weight.

Many formulas have been proposed, but the only formulas in anything like general use are known as the Engineering News Formulas. They are:

For a pile driven with a drop hammer $P = \frac{2\,W\,h}{S+1}$

For a pile driven with a steam hammer $P = \frac{2\,W\,h}{S+0.1}$

to which P is the safe load in pounds, W the weight of the hammer in pounds, h the fall of the hammer in feet, and S the penetration or sinking in inches under the last blow, assumed to be at an approximately uniform rate. They are deduced for wood piles; but are the best there are for concrete piles.

Mörsch gives the Brix formula as usually employed on the Continent:

$$p = \frac{h\,Q^2\,g}{2e\,(Q+g)^2}$$

wherein h is the fall of the hammer;

Q the weight of the hammer;

g the weight of the pile;

e the penetration of the pile under the last blow;

p is double the safe allowable load for the pile.

The quantity e will naturally be the average of the last few blows.

CHAPTER XV

ELEMENTS OF ECONOMIC CONSTRUCTION AND COST OF REINFORCED CONCRETE WORK

1. Introductory. Reinforced concrete construction of buildings presents problems which from an economical standpoint are so complicated that they cannot well be investigated as questions of maxima and minima by means of equations which show the cost of the several variable items and determine their proper relation by mathematical treatment. For this reason it is necessary to pursue the investigation along simpler lines, by taking up questions of general arrangement, column spacing, spacing of beams, choice of type of reinforcement, cost of centering for each kind, cost of aggregate, etc.

2. Column Spacing. Eighteen feet center to center of columns in each direction usually costs less than for shorter spans while the increase in cost with increase of column spacing up to twenty feet is very little, if the loads are heavy, and the building high. In wholesale hardware buildings, the customary requirement is that of lines of shelving and boxing about twelve feet centers. For such a building, diagonal spacing of columns with flat slab construction, making the columns about seventeen feet between centers, will provide the desired spacing in a multiple of twelve feet for the boxing and shelving, i. e. twenty-four feet in directions parallel to the sides of the building.

3. Floors. In treating the strength of floors by comparison on the principle of proportion it has been noted that the coefficient of bending with continuous flat slab type is smaller than the corresponding coefficient for beam construction. This advantage is offset for light loads by the disadvantage of the smallness of depth which is a factor in the moment of resistance. Further, stiffness varies as the square of the depth, but the coefficient for deflection is smaller than for simple beams. Hence it is evident that for stiffness under light loads, lower percentages of steel are requisite than in the beam type of construction. Where the loads are heavy the continuous flat slab type becomes more and more economical compared with other types of floor as the depth of slab increases by reason of the relatively small coefficient of bending. A little computation as to the cost of concrete and steel shows that considerable variation from the exact economical proportions of the two materials for a given strength will make comparatively little difference in the ultimate cost, whereas in the flat slab types using a smaller amount of steel and more concrete does make a large increase in the stiffness.

Flat slab types have the advantage of simpler centering and lower cost of placing the steel, but aside from that they possess little advantage over the two-way beam system for light loads, tho requiring less material both of concrete and steel for heavy loads and moderate spans up to twenty-five or thirty feet, and they compare favorably for greater spans where the loads exceed five hundred pounds per square foot.

Slabs reinforced in one direction, being supported on two sides only, are at a disadvantage, since the coefficient of bending is three times as high with one-way reinforcement, as it is for the slab reinforced in two ways and supported on four sides.

In comparing Types III and IV, it should be observed that if the beams of Type III are of twice the thickness of the slab of Type IV, the same weight of reinforcement is required in the beams of Type III that is required in the whole floor of Type IV.

Usual proportions would be a depth of beam equal to three times the thickness of the slab of Type IV requiring two thirds the steel for the beams.

The apparent moment to be resisted in the slab supported on four sides Type III is but two thirds that to be borne by slab of Type IV. Hence a slab may be used of less thickness than with Type IV tending to equalize the concrete quantities and leaving the difference in cost of forms and steel to be considered.

Type II is frequently made of a combination of hollow tiles and thin reinforced concrete beams between the hollow tiles, in order to secure economic depth and reduced weight for long spans, with decreased deflection under load. Where this type is used, the cost of centering is kept down to a figure approaching that of the flat slab types but extra care is required in placing the steel and the labor of putting tile in position and the cost of the tile is added. Where tile is low in cost and concrete material is high in cost, tile and concrete for a light building may be more economical. On the other hand, the risk of erection is greater with this type on account of the brittleness of tile and the character of failure of one way reinforced slabs as pointed out in the discussion of beams. For heavy loads, however, this type cannot compete with the natural concrete types, III and IV.

4. Centering. As we have noted, centering is one of the important considerations since the cost of centering runs to twenty-five to thirty-five percent of the total cost of the ordinary floor.

In the selection of the type of floor to be used, the cost of centering is so large an item that it should be given careful consideration in any approximate solution of an economic design.

Where the spans are long and the loads are light and the cost of concrete materials, stone, gravel and cement are very high, flat slab types cannot compete with the beam type. Type III as a beam and slab type is more economical than other beam types for heavy loads and panels of sixteen to eighteen feet. Where the columns are equally spaced, the economy is greatest. Where the spacing is such that the panels are rectangular and one side is less than six-tenths of the longer side, its economy disappears and it is preferable to use an intermediate beam, thus dividing the slab into panels more nearly square.

For joist and girder construction surmounted by light slabs, the spacing of the ribs is governed by the character of the centering used. If the centering is arranged in panels so that it is easily handled as such, six to ten foot spacing of beam joists works out very economically. Considerable increase, however in the cost of centering is brought about by the additional framing of the joists to the main beams or girders.

No exact rule can be given covering all cases, as the conditions of the problem, such as size of the building, manner in which it divides up for the purposes intended, etc., fix so many of the conditions that any rule disregarding these conditions might readily be misleading. The number of stories figures largely in the cost of centering. The cost of framing beam boxes as a rule must be figured for three stories. Additional stories, if the framing is worked out so that the same boxes can be used over and over again reduces the cost per foot greatly and these are conditions which must be taken into consideration in any practical comparison of different types.

5. Columns. The hooped column is the safest type to erect and where the loads are at all heavy the most economical type to adopt. The economy of the reinforcement depends largely upon the adoption of the proper proportion of hooping and vertical steel. The hooping adds to the toughness of the column and to its ultimate strength but does not raise the point where the column shell commences to scale and chip unless the hooping be combined with the proper proportion of vertical steel. When this has been done, very high values indeed can be safely developed. The carrying capacity, however, of the column is secured at a minimum cost by concrete rather than by reinforcement. Still a certain amount of steel is necessary to secure toughness and resist flexure. Again, consideration of the value of floor space frequently limits the size of the column which the architect or owner is willing the designer should employ.

As to what may be done with the reinforced concrete column, Turner has used a twenty-seven inch core, heavily banded and reinforced vertically, for working loads from eleven to twelve hundred tons. This pressure being a little over 4000 pounds per square inch of core area. The development of such high working values by heavy reinforcement, tho unobjectionable from the standpoint of safety and desirable from the standpoint of occupying small floor space is not economical from the standpoint of first cost of providing a post of proper capacity for a given load. A rich concrete is more economical than a lean mixture, and where the loads are heavy a 1 : 1½ : 3 mixture is recommended in place of the usual 1 : 2 : 4 employed under ordinary conditions.

6. Bearing Walls or Full Concrete Skeleton. A very important question in economic design is the question as to whether bearing walls are to be preferred to a full concrete skeleton.

For such a low building as four to five stories, bearing walls are generally cheaper than a full concrete skeleton. For buildings higher than five stories a full concrete skeleton with curtain walls costs less than heavy bearing walls.

In putting up a concrete skeleton, the additional cost involved in making provision for two or three additional stories is generally so small that it is advisable for the owner to make this provision if there is a reasonable probability that he may use the additional floor space in the future and the value of the real estate warrants such investment.

7. Concrete, or Brick Exterior Walls. In some cases where the concrete aggregate is cheap and union brick layers' wages are high it is better to use a concrete exterior wall. Generally, however, exterior walls may be constructed much more cheaply of brick, or of some material that can be laid up without the necessity of using forms, since the forms for exterior wall construction run into money quite rapidly.

8. Rich Mixture. Economic construction in reinforced concrete requires a rich mixture. This is a necessity, first, from the standpoint of certainty of computation, second from the standpoint of quick hardening which enables the early removal of the forms with economy incident to repeated use of the same form lumber, and from the standpoint of economy due to the fact that we can use less material of good quality which we can absolutely depend upon than we can of material of an inferior quality and uncertain character which is liable to be discredited by reason of its slow hardening through the lack of necessary amount of cement. Further, where the material used has been of good quality the construction can be increased in strength to any desired degree by the addition of more good concrete tho the strength so secured will not be at so low a cost as if the original design was for heavier construction.

9. Economy in Selecting Aggregate. Good bank gravel when obtainable makes an excellent aggregate. Its adaptability for the purpose should be determined by screening out the sand and pebbles which are under ⅛ inch diameter and comparing the volume of sand with the volume of coarse aggregate. If these proportions depart from those desired, that is, one cement, two sand, and four of the coarse aggregate, then the cement content must be increased to make the mortar of the proper proportion, or else crushed stone must be added to the mixture.

Where the expense of securing crushed stone is high and the cost of cement is low, the addition of more cement to keep the mortar a true 1 : 2 mix is cheaper than the addition of stone, and in a way preferable, because with the excess of mortar there is less liability of voids and poor work. Sometimes crushed stone and gravel is not available and a good hard furnace or smelter slag may be secured. Slag should be examined for chemical impurities which might injure the cement and for hardness which determines its fitness as a good aggregate.

10. Cinders. Cinders are sometimes used as an aggregate for concrete. Cinder from the soft Iowa coal is generally very injurious to the cement. In fact it may be stated as a general rule that the only cinder fit to make a permanent concrete is that which is a hard or more or less vitrified clinker such as generally results from burning soft coal with a mechanical stoker. Too great care cannot be exercised in this respect as upon the character of the aggregate and its freedom from sulphur or other injurious chemical elements which would damage the cement, depends the permanence and integrity of the work.

In general a clinker concrete should not be used where a high degree of strength is required. It is desirable to use it for such work as roof work where the spans are short and it is desired to nail a tile or slate cover to the concrete roof slab. For such purposes the concrete should not be mixed too rich, otherwise it will be difficult to nail into it.

11. Adaptability. Reinforced concrete is not adapted for long spans and light loads. For instance, in a span of fifty or sixty feet in a shop or factory building having only a light roof load, reinforced concrete is not an economical material to use. Structural steel costs far less and is generally employed. For bridges of long spans and light loads reinforced concrete is not economical in first cost. Where the loads are heavy, as for a city bridge, and the grade such that there is opportunity for ample rise, a reinforced concrete arch may be built at a cost not greatly exceeding that of a good structural steel bridge and when the maintenance charges are considered the concrete will be the least expensive.

For office buildings and ordinary business blocks reinforced concrete will in general save the owner from one-half to three-quarters of the cost of a structural steel skeleton.

Reinforced concrete is particularly well adapted for school buildings. The difference in cost between the ordinary timber floor and a reinforced concrete floor will frequently not exceed ten or fifteen cents per square foot. With this fact in mind it is really astonishing to observe how frequently dangerous fire traps are erected to serve as school buildings on which expensive and ornamental exteriors have been used when plainer buildings with fireproof construction could be erected for the same money.

The archietect for a school building, in order to make a show, frequently specifies a fancy brick exterior, terra cotta or stone trimmings and other external frills and then economizes in the interior construction of the building by the use of timber joists $1\frac{1}{2}$ or $1\frac{3}{8}$ inches thick by 16″ covered with $\frac{7}{8}$ inch rough floor and $\frac{7}{8}$ inch hardwood finished floor, with wood lath and plaster on the under side, electric wires, heating flues, etc., between the joists. This is a construction in which if a fire once started there would be hardly time for the occupants of the building to escape before the collapse of the floor with the probable loss of life incident thereto.

The Collinwood disaster well illustrates this fact. The state of Wisconsin has passed a law making it compulsory to build school buildings of fireproof materials and other states may well follow her example.

For buildings subject to the vibration of heavy machinery concrete steel construction has many advantages. Properly designed, the joints (connections of floor to columns) are far more rigid than in any of the old types of construction, hence a rigid building costs least in concrete steel.

The Forman-Ford Company, plate glass dealers, etc., make the statement that twenty-five percent more work is done in their cutting and polishing department in their new concrete building with the same men than in the old timber framed structure previously occupied owing to the increase in rigidity of the structure.

Impact or shock at any point of a steel structure is propagated longitudinally along elastic members extending in a linear direction from the point and it goes practically undiminished to the far ends of these members where it is subdivided among other members and propagated still further. An allowance up to eighty or ninety percent is usually added for impact to the static effect of a moving load in bridges.

Impact, or the dynamic effect upon any point of a reinforced concrete slab, however, is entirely different from this. In the first place, the effect of the blow does not travel in one direction only but in all directions radially from its point of application, so that in a very thin slab its effect at any other point would be inversely as the distance and in a very thick slab inversely as the square of the distance. This would make the allowance for impact in the thick concrete floor of a bridge or building very small in comparison with that inevitable in steel construction.

Secondly, the effect of impact must be inversely proportioned to the weight of the body receiving the blow. Now in monolithic concrete construction the mass affected is of far greater weight than would be the case in a steel frame which is made up of independent steel members and tile as in the old style buildings and for that reason the effect of impact on a monolithic concrete floor would be reduced by a large percentage.

Third, the continuity and stiffness of the floor greatly reduces its vertical, lateral and torsional deformations below those of the steel structure. The work done during an impact and its effect depend on the amplitudes of the deformations. In particular the horizontal resistance of the slab is many thousand times that of the steel members in a structure. The vibratory energy absorbed by the slab during impact is consequently small.

Fourth, the small amount of energy which is absorbed is not transmitted (as it is in a highly elastic and resilient structure) to a considerable distance in the slab, but owing to the nature of the concrete, is dissipated near its source, transformed into heat, and rapidly absorbed.

Fifth, concrete slabs are tough and not brittle, like terra cotta, for example, so that, in cases where great weights have fallen on them, little effect has been produced, whereas brittle slabs such as those of concrete and tile have been smashed under such circumstances, and have failed.

The concrete of the compression zone is such a shock-absorber as to protect the tension zone from jarring and vibration, both as regards steel, in tension and the bond of the concrete to the steel as well.

For all these reasons the shock which a rolling load imparts to a slab is inconsiderable, and is absorbed and dissipated so readily that it is a negligible factor, rendering reinforced concrete adaptable for use in railroad structures and in buildings where the service is most severe from shock of machinery such as beater floors in paper mills and hammers in stamping and working metal.

As illustrating these statements, an interesting accident occurred in Winnipeg where a heavy cornice block of hard Tyndale limestone, weighing between a quarter and a half ton, was dropped seventy feet, striking a 5½ inch concrete slab 19 feet square in the center. The effect of this blow was a small dent half inch deep where the corner of the stone struck the slab and the stone itself was badly broken and shattered. The slab, however, was uninjured and showed no cracks or evidence of over-strain.

In Minneapolis, a steel water-tower failed under wind pressure and a fifty ton tank dropped ten feet on a light roof slab of concrete reinforced after the manner of the Mushroom system in four directions. The slab was approximately 22 feet by 23 feet clear span, seven inches thick, reinforced with 7/16 inch rounds averaging eight inches centers in two directions with diagonal belts of fourteen

7 /16 inch rounds. The shock of the fifty tons falling ten feet dented the slab down about eight inches and produced some shear cracks but it carried the load and caused the contents of the tank to fall outside of the building, demolishing two freight cars and the awning over the loading platform. The toughness of this slab saved a wreck of the side of the building and the elevator machinery.

Other interesting cases might be cited, but it is thought that these unusual examples illustrate the remarkable toughness and dependability of concrete reinforced in multiple directions and its great adaptability from the standpoint of capacity to withstand severe usage with high degree of safety for all purposes when scientifically designed and properly executed.

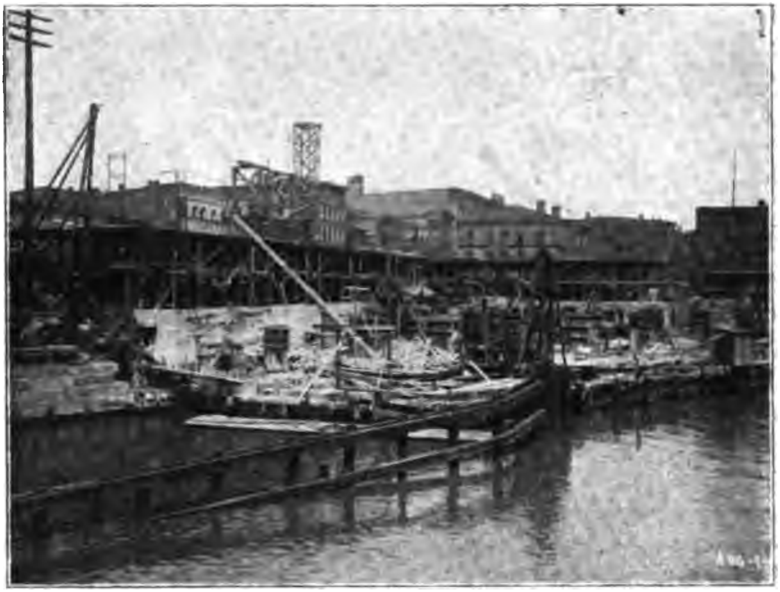
12. Rapidity of Erection and Ease of Securing Material. No type of building can be so rapidly or quickly designed, detailed and erected, as the natural types of reinforced concrete construction. If we take Type IV, for instance, a single computation is sufficient for a panel and where panels are tabulated for various loads two hours' work is sufficient to make the computation for a given size of factory building or manufacturing plant, including an estimate of the cost of reinforcement, quantities of concrete and centering with sufficient precision for bidding purposes.

In no type of building construction can the materials be secured so promptly as for the reinforced concrete structure. An ordinary four or five-story building can sometimes be erected complete in the time ordinarily required to get out the shop details for a structural steel frame. Especially where the building is irregular in form there is this advantage with reinforced concrete, that the joints are made with the cement in plastic form, that the rods can be lapped more or less over the supports, avoiding the necessity for the large amount of figuring required for the skew connections of structural work, hence the engineer's end in this line of building construction is greatly simplified.

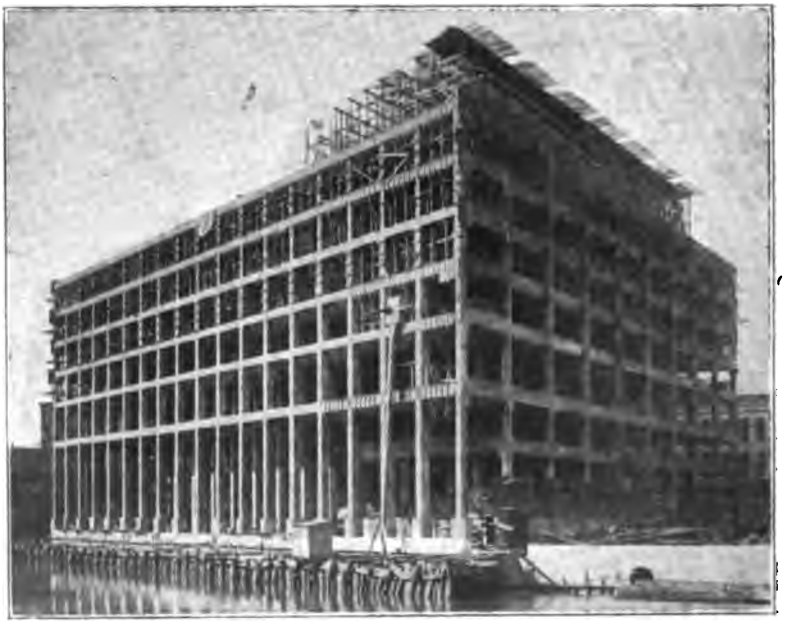
The accompanying figure shows the rapidity of construction of the Bostwick Braun Building; its condition August 1st showing the sea wall and adjacent footings incomplete; November 11th, showing the centering nearly completed for the roof, or eight stories in place.

This is merely a usual example in the construction of a large building. A floor per week of the concrete skeleton and rough slab can easily be erected under favorable weather conditions.

There is little difference between a large building and a small building in this respect, since with a larger building it is possible to rig up in



Aug. 1st, Showing Seawall and Adjacent Footings Incomplete.



Nov. 11th, Showing Centering Nearly Complete for the Roof, Eight Stories in Place.

a manner that will facilitate the handling the work more rapidly. In fact where there is a large area it permits the employment of more men and makes it possible to keep them at work on the various features such as centering, pouring concrete, placing steel, in one continuous operation, centering going ahead for one floor as the steel is being placed, the concrete men in turn following up those placing the steel. When the carpenters are thru with this floor they immediately proceed to erect the forms for the next on that portion of the floor where the concrete has been cast.

Where the amount of work to be handled runs eight to ten thousand yards, it pays well to rig up with overhead bins and mixing plant and to arrange for the use of half-yard dump cars in placing the concrete. Where the yardage is much less the wheelbarrow or two-wheeled truck and scale hoist becomes an economic method of handling material. The later mixers are arranged with a charging device which saves wheeling material up an incline as was customarily done in earlier work.

13. Analysis of Items of Cost. In arriving at a detail estimate of cost we have the following items to consider:

Basis of Labor

Materials and cost of handling:

Concrete, Unit price	Cement		Quantities and base prices.
	Sand	or gravel	
	Stone		
	Water		
	Common labor		
	Cost of plant		
Steel	Cost of metal		
	Cost of unloading		
	Labor, cost of bending (union or common)		
	Labor, cost of placing (union or common)		
Centering	Cost of lumber		
	Cost of framing beam boxes, columns, etc.		
	Cost of erecting and rehandling		
	Slab forms, beam forms and column forms.		

Season of the year.
Floor finish or strip fill.
Dead expense.
General data on costs per foot of floor and items entering into it.

14. Labor, Unit Prices, Quantities of Material. Under the general heading of labor the contractor must consider, first, the wages per hour; second, the character and efficiency of the labor, whether the labor is union or non-union, probability of strikes and delay of work into the unfavorable season when artificial heat must be used.

Where trade unions are strong the specialist in reinforced concrete can never tell when his work will be tied up by some disagreement between master plumbers and walking delegates or other trades with which he has no relation whatever further than that a sympathetic strike may be called without notice or grievance at any time and his operations brought to a standstill.

This condition means idle equipment and sometimes cost of heating materials and may mean readily an additional cost of five to ten per cent in the work.

In Chicago the bricklayers' union demands that the contractor shall keep employed an extra brick foreman who is supposed merely to watch the placing of concrete wherever there may be brick work on the job.

While a labor union should prove of benefit to employer and employee alike in case its motto is efficiency and skilled service, it loses both public sympathy and support when it cultivates inefficiency and loads the work up with men who are useless as is the case with an extra foreman. In many cases concrete is used for exterior walls in place of brick where, were it not for this short-sighted policy, brick would be used from an economic standpoint.

For purpose of discussion and comparison we will take the following costs: Common labor, $2.25 per day of ten hours; carpenters, $3.00 per day of eight hours; steel to be placed by common labor at $2.25 to $3.00 per day.

Unit Price of Concrete

1 : 2 : 4 Mix

Material for one cubic yard of concrete, wet mix:

Cement	1½ bbls.
Crushed stone	.9 cubic yards.
Sand	.45 cubic yards.

Where a crusher run, including dust of good hard crystalline stone is used the sand may be readily reduced to one-third yard.

1 : 1½ : 3 Mix

Material for one cubic yard of concrete, wet mix:

Cement	2 bbls.
Sand	.43 cubic yards.
Stone	.85 cubic yards.

1 : 3 : 5 Mix

Material for one cubic yard of concrete, wet mix:

Cement	4½ sacks = 1.125 bbls.
Sand	.52 cubic yards.
Stone	.85 cubic yards.

The above amounts required to make a cubic yard of wet mixed concrete in place may vary somewhat with the character of the crushed stone or gravel, but for estimating purposes they are conservative.

Labor of Handling

Given an ordinary equipment such as a half-yard Smith, Cube or Ransome machine the labor cost of handling concrete may be stated as follows:

Wheelbarrow gang, from $1.25 to $1.50 per cubic yard, including cost of coal or gasoline for the engine. In walls, footings or where there is quite a mass this may be reduced to $1.00 per cubic yard.

On a large job where one-half yard cars are used and overhead bins for handling the aggregate by gravity, the labor cost may be reduced to 35c to 40c per cubic yard. To this must be added, however, the cost of fitting up the plant which will increase this figure to sixty or even seventy cents per cubic yard.

The labor costs will increase or decrease as the price of common labor is above or below twenty-two and one-half cents per hour, figured upon.

Cost of cement varies with the market and distance of the work from the nearest mill from eighty cents or a dollar per barrel to two or three dollars.

Cost of crushed stone varies with the locality and distance of the work from railroad or crushing plant.

In Minneapolis and St. Paul, from $1.25 to $1.75 on the work. Milwaukee, $1.25 to $1.59. Washed gravel, Ohio river points, $1.00 to $1.25.

Cost of carting and hauling must be investigated in each individual case. In many of the smaller towns good concrete gravel can be secured as low as thirty to fifty cents per cubic yard and in order to give a clear idea as to the general questions of cost these variables must be carefully considered and investigated by the bidder if figuring reasonably close.

In securing this essential information the conservative business man will secure quotations in writing, especially when not personally acquainted with the reliability of the parties quoting; then if the work is secured he may at his option hold the bidder to his price or seek redress by suit.

In giving the foregoing average values it should be noted that the cost of placing varies eight to ten per cent with the character of the reinforcement. Where there are numerous beam boxes and stirrups and increased work of puddling the concrete, the cost may readily run five or six per cent above the average while where there is a plain flat slab such as the Mushroom system the cost will readily run five or six per cent lower than the average given.

15. Cost of Steel. Medium Steel, Open Heath or Bessemer, Manufacturers' Standard specification is at present writing at $2.35 base, Pittsburg.

The base price is given in all the engineering and iron trade papers. All bars from ¾″ rounds to three inches are base. Smaller bars are sold at base plus card extras.

The following is the standard steel classification:

	Extra	
3/4″ to 3″	Base	Rounds or square.
5/8″ to 11/16″	.05	
1/2″ to 9/16″	.10	
7/16″	.20	
3/8″	.25	
5/16″	.30	
1/4″ to 9/32″	.35	
1 to 6″x3/8 to 1″	Base	Flats and heavy bands.
1 to 6″x1/4 to 5/16″	.20	

The above are full extras and such sizes as we can ordinarily use to advantage in concrete steel construction.

In figuring, take base price plus extras plus freight to destination. Freight rates to all points in the United States and the Dominion of Canada are given in compact form in a book published by the American Steel & Wire Company.

Cost of deformed bars rolled to standard specification at the present writing, one dollar per ton above plain bars. Special reinforcement sold with design from eight to twelve dollars per ton additional.

16. Cost of Bending. Medium steel with proper equipment rods for the Mushroom system can be bent cold for fifty to sixty cents per ton; where high carbon steel is used and rods are heated, one and a half to two dollars per ton for bending.

Beam rods, such as are used in Turner beam system, can be bent for from two to two and a half dollars per ton; where more complicated bends are employed, from two and a half to three and a half per ton.

Cost of placing steel, including handling and bending, in Mushroom flat slab work has run from six to ten dollars per ton. With a beam system in beams spaced four to six feet centers a cost of ten to twelve dollars would be a fair basis upon which to figure.

17. Cost of Hooping for Columns. Spirals made at the shop can, at the present time, be furnished at a less cost than they can be fabricated in the field, except in parts of Canada where this statement may not be true. Shop work is better done than field work and should be preferred other things being equal.

18. Cost of Centering. No item in concrete construction is so generally underestimated as the cost of false work for reinforced concrete. In fact, so generally is this the case that the contractor inexperienced in this kind of work is more than likely to underbid those possessing both equipment and experience by reason of underestimating this item of cost.

Cost of the centering per foot of floor, including columns and beams, will vary anywhere from six to more than twenty cents per square foot, depending on the following items:

1. The number of beam boxes, whether they frame into each other or into column boxes only.
2. The number of columns for a given floor area.
3. The number of stories or floors that are alike.
4. The rapidity with which it is desired to push the work and whether the weather conditions are favorable for the prompt removal of the forms.

Where the building has a full concrete skeleton, centering costs generally are a third more than where bearing walls are used. Evidently the greater the number of stories the more times the lumber may be moved up and used over. Where the work is to be rushed rapidly in cold weather a larger amount of lumber is required.

The practical constructor is inclined to check his estimate of cost on the basis of so much per foot of floor for centering for reinforcement and concrete, and estimating in this rough way will generally detect any error of more than four or five per cent in an elaborate detailed estimate.

Attention may be called to the following elements necessary in any estimate:

1. Lumber required, nails and fastenings.
2. Carpenter labor of framing beam boxes, column boxes, etc., per thousand feet.
3. Labor of setting plain slab forms.
4. Labor of taking down forms and moving up to upper story per thousand feet B. M.
5. Waste of lumber and value of old centering.

Under 1, the amount of lumber required, it should be observed that the amount will vary with the type of design. In such a type as the Mushroom system, Type IV, Fig. 14, there must be for the sheathing approximately one foot B. M. for each flat foot of floor area. For the joist, $\frac{3}{4}$ of a foot B. M. per flat foot of floor area. For ledgers, one-third of a foot, B. M. for each flat foot of floor. For uprights, $\frac{5}{8}$ of a foot B. M. for each flat foot of floor. For columns, spacing 18′ centers, from one-third to one-half foot B. M. for each flat foot of floor. Total, about three and one-quarter or three and one-half feet B. M. per foot of floor.

If the work is to be pushed rapidly it is necessary to figure, under favorable conditions for centering, not less than two complete floors of centering plus waste. If the weather conditions are unfavorable there should be enough lumber for centering for three to four floors. On a building having eight stories we would ordinarily figure enough centering for three floors, plus waste. With the flat slab system there is no waste with the joists as they are simply lapped by and the waste in the boards would amount to about two per cent each time they are used. There will be some waste in the uprights if the stories are of different heights, which must be figured in each individual case.

Where a beam system is used the waste will be much greater as the loss from breakage and cutting the lumber to the size of the panels will generally run the waste up to ten to twenty per cent of the lumber in each floor, and sometimes much more than this. Also the surface contact is increased by the area of the sides of all beams requiring additional lumber.

Cost of Framing. Labor for framing beam boxes, column boxes, etc., will generally run about twelve dollars per thousand feet B. M. Labor of placing plain slab forms, carpenter's wages, being figured at 37½c per hour, will run about five to six dollars per thousand feet. The cost of taking down the forms and moving them up should run about three dollars per one thousand feet B. M., for the flat slab type and five to seven dollars per thousand where there are a large number of beam boxes, etc. Nails and fastenings are generally a small item.

Where sheet metal is used for the sheathing the cost per foot of laying it and greasing it with paraffine is about one-third the cost of placing boards, altho the first cost of the metal is considerably higher.

Mr. L. C. Wason, president of the Alberthaw Construction Company, of Boston, at the fifth annual convention of the National Association of Cement Users at Cleveland, Ohio, presented a paper on costs from which the following table is condensed, giving the cost of handling and some very interesting costs of centering. It would be well for the reader to look up this paper which is reprinted in part in the Engineering News, January 14, 1909, and a number of the other engineering papers.

The following table, condensed by the Engineering News, from the original paper, is given as a fair indication of the variation in cost of different designs and different conditions. The author states that only typical cases are given where the items of cost were accurately known. Enough are given for a fair average except in the case of long span flat slab which appears to him by comparison a recent type of construction.

By reference to the general averages on form work in the accompanying tables the cost of forms per square foot of surface contact, namely: Columns, $0.13; floors with reinforced concrete beams, $0.116; flat floors without beams, $0.111; short span slabs between steel beams including the fireproofing on the side of the beams, $0.05; walls exposed to view above ground, $0.093; the writer believes are all higher in price than usually believed to be a fair cost by most builders. It is upon the success of handling forms that good results financially depend. In regard to concrete, labor is

TABLE 1.—SHOWING COST OF FORMS AND CONCRETE ON VARIOUS MEMBERS IN REINFORCED-CONCRETE STRUCTURES

	Forms per sq. ft.				Concrete per cu. ft.						
Location	Carpenter Labor	Lumber	Nails and Wire	Total	Concrete Labor	General Labor	Cement	Aggregate	Team and Misc.	Plant	Total
PLAIN CONCRETE COLUMNS											
Office building, Portland, Me.	$.133	$.039	$.011	$.173	$.064	$.004	$.087	$.084	$.012	$.022	$.273
Coal pocket, Lawrence, Miss.	.057	.024	.001	.082	.166	.003	.073	.041	.008	.016	.307
Mill, Southbridge, Mass	.097	.082	.002	.181	.073	.056	.107	.035	.027	.030	.328
Mill, Attleboro, Mass	.093	.022	.001	.116	.110	.014	.062	.038	.013	.034	.271
Mill, Southbridge, Mass	.080	.056	.001	.137	.108	.048	.100	.037	.013	.034	.340
Coal pocket, Hartford Ct	.098	.047	.002	.147	.089	.043	.069	.055	.017	.013	.286
Garage, Brookline, Mass	.071	.051	.002	.124	.070	.028	.072	.058	.041	.020	.289
Warehouse, Portland, Me	.118	.016	.001	.135	.087	.027	.087	.070	.039	.025	.335
Textile mill, Lawrence, Miss.	.061	.013	.001	.075	.095	.019	.109	.027	.018	.015	.283
Highest	.133	.082	.002	.181	.166	.056	.109	.084	.041	.034	.340
Lowest	.057	.013	.001	.075	.064	.003	.062	.027	.008	.013	.271
Average of 9	.082	.036	.001	.130	.096	.027	.085	.049	.021	.023	.301
REINFORCED-CONCRETE BEAM FLOORS											
Highest	.165	.107	.004	.275	.186	.035	.194	.101	.052	.055	.470
Lowest	.037	.027	.001	.067	.047	.004	.071	.037	.007	.010	.202
Average of 18	.070	.045	.002	.116	.111	.020	.106	.063	.025	.024	.354
FLAT SLAB FLOORS											
Highest	.078	.039	.003	.118	.146	.017	.109	.084	.026	.039	.374
Lowest	.067	.037	.001	.106	.043	.004	.087	.053	.012	.010	.252
Average of 3	.071	.038	.002	.111	.097	.009	.096	.070	.019	.024	.315
REINFORCED-CONCRETE SLABS BETWEEN STEEL BEAMS											
Highest	.110	.071	.003	.184	.144	.048	.208	.080	.064	.046	.428
Lowest	.028	.012	.001	.049	.073	.005	.076	.026	.004	.010	.272
Average of 13	.061	.032	.002	.095	.102	.019	.128	.068	.024	.017	.359
BUILDING WALLS ABOVE GRADE											
Highest	.136	.073	.005	.176	.146	.052	.105	.187	.077	.055	.446
Lowest	.046	.016	.001	.079	.042	.004	.034	.043	.007	.005	.174
Average of 17	.085	.036	.002	.128	.090	.016	.073	.076	.025	.019	.301
FOUNDATION WALLS											
Highest	.134	.048	.004	.193	.213	.037	.203	.116	.057	.040	.599
Lowest	.032	.009	.001	.056	.040	.002	.038	.027	.003	.010	.148
Average of 14	.068	.033	.002	.103	.076	.015	.080	.062	.019	.017	.269
FOOTINGS AND MASS FOUNDATIONS											
Highest	.119	.077	.003	.198	.081	.020	.098	.099	.013	.049	.275
Lowest	.016	.006	.001	.018	.025	.001	.047	.043	.003	.010	.181
Average of 10	.057	.034	.002	.093	.045	.007	.071	.077	.007	.021	.229

the variable item which must be carefully considered. Any person of intelligence can make a careful estimate of the materials to be used, but note the average prices of labor per cubic foot of concrete, namely: For columns, $0.123; beam floors, $0.131; flat floors, $0.106; floors between steel beams, $0.121; walls, $0.106; foundations, $0.091; and mass work in connection with buildings, $0.052. Not until the last item is a price reached which according to observation and experience must be expected to obtain in ordinary building work. Many who have had wide experience in handling large quantities of concrete in mass have occasionally attempted a lighter type of construction and have been greatly surprised at the large expense connected therewith. Men with this experience have frequently added fifty to one hundred per cent to the cost of mass work and only by doing so have they felt that they were sufficiently covered for light structural work.

Table II is an exact copy of a "master card" which gives the complete financial history of the job, when it is finally completed. The first column, which is blank, is occasionally used for an estimate of the first cost, the proposal including the profit as well as the estimated actual cost. It will be seen that on some items, a loss was incurred, as well as a profit on others, showing that it is difficult to reach the right price on everything, even on work on which a company is fairly experienced, and also that when slight changes are made by the owner or architect they often entail heavy loss even though the changes appear to be extremely trivial. Take the case of the external walls. The owners furnished the window frames and sash, which were all of metal. The original design was for a frame with two sash, which could easily be put into a six-inch wall.

TABLE II.—TYPICAL "MASTER CARD,"

Job No. 747. Date, May 24th, 1906. Mill, Tappan Bros., Attleboro, Mass.

	Proposal	Actual Cost	Per cu. ft.	Profit	Loss	%
Total	$35,164.55	$31,330.48	....	$3,834.07	....	11
Excavate	790.00	823.18	$0.021	....	$33.18	..
Footings and Fn	1,738.00	1,033.57	.137	704.43	....	..
			Per sq. ft.			
Exterior walls	1,955.00	2,162.02	.190	....	207.02	..
Wall and Fn. centers	1,520.00	3,630.08	.125	....	2,110.08	..
Floors 6½ ins. thick	8,883.00	6,544.16	.339	2,338.84	....	..
Roof 5½ ins. thick	2,869.00	1,713.51	.237	1,155.49	....	..
			Per lin. ft.			
Columns, 20 ins. x 20 ins	832.00	676.65	1.470	155.35	....	..
Stairs	883.00	910.35	.912	....	27.35	..
			Per sq. ft.			
Tool surface	469.00	636.53	.056	....	167.53	..
Ornaments and cornice	348.00	164.33	....	183.67	....	..
Ventilators on roof	44.00	35.64	.....	8.36	....	..
			Each			
Set windows and door frames	852.00	929.99	2.19	122.01	....	..
			Per sq. ft.			
Interior partitions	1,770.25	1,656.35	.189	113.90	....	..
Bolts and iron work	253.00	257.06	....	....	4.06	..
Stair railing and grill	387.00	654.00	.....	....	267.00	..
			Per M.			
Screeds and settings	1,086.00	835.12	52.17	250.88	....	..
2-in. Spr. plank and laying	2,839.00	1,431.69	33.30	1,40731	....	..
⅞-in. maple	1,738.00	1,788.88	89.44	....	50.88	..
Motor shaft	379.50	533.19	98.89	....	153.69	..
Motor shaft found	98.00	70.07	.....	27.93	....	..
Roofing and conductors	1,255.00	1,026.06	.....	228.94	....	..
			Per sq. ft.			
Paving	1,009.00	747.54	.094	361.46	....	..
Retaining wall centers, per sq. ft	....	....	.211	....	....	..
Retaining wall, concrete per c. ft	429.00	316.90	.175	112.10	....	..
Painting	400.00	375.00	.....	25.00	....	..
Steel footings and walls	300.00	218.91	.....	81.09	....	..
Plant, frt., etc	1,860.00	2,271.73	.....	....	411.73	..
Bond	100.00	120.00	.....	....	20.00	..
Extras	77.80	67.97	.....	9.83	....	..

They later decided, for greater fire protection, to use four sash. This required an eight-inch wall instead of a six inch, and the form work on the inside had to be built inward and then the space under the windows paneled to save material. To save making a very narrow panel at the side of the window, which would cost more than the concrete saved, the space was filled up solid so that the columns appear to be wider than they were actually figured. This slight change, which did not appear great at the time, when the job was entirely complete showed that the concrete of the walls showed an actual loss instead of profit because the form work cost more than twice what was originally estimated that it should cost.

19. Season of Year. The season of the year has to be considered in its relation to the cost of reinforced concrete work. In the summer season when the concrete dries out rapidly the forms may be removed every ten to twelve days, while in the fall and early spring during frosty weather the water must be heated or the forms left in place longer, requiring more lumber for centering. In the winter when the materials must be heated by artificial heat and artificial heat used in sweating out the concrete, the cost of work will be increased from ten to twelve per cent. Additional cost of merely heating the water is of course small. In the chilly weather of fall or spring good results may be frequently obtained merely by turning the exhaust steam into the water barrel and warming the water up so that the concrete will set quickly notwithstanding the chilly temperature.

20. Dead Charges. No contracting firm can do business without a considerable general expense, which must be distributed over all work executed by them. This expense includes office expense, advertising, soliciting work, estimates on not only the work taken but the work which the concern fails to secure, depreciation of the plant, freight, storage and equipment, the cost of keeping the organization together in slack periods. This expense may readily vary with various concerns from five to seven per cent of the cost of the work executed. In addition to this dead expense and the actual cost of labor there must be included the item for liability insurance which the contractor cannot afford to neglect to carry. Frequently the owner requires a surety bond for the faithful execution of the work and the payment of bills, the cost of which must be added to the incidental charges in the estimate of cost.

21. General Data on Cost. The architect is in the habit of figuring the building as so much per cubic foot. For heavy warehouses with the plainest kind of finish and large size the cost per cubic foot may run as low as six and one-half to seven cents up to ten and twelve cents for the smaller size of buildings with office fixtures, plumbing and the like. No approximate cost per cubic foot of any value can be given for office buildings, hotels and the like, since this item would vary greatly with the character and difference in the quality of the finish, fittings and the like.

For the concrete end of the building, however, a rough approximate estimate can be made very readily by figuring a unit price per square foot of floor area. In a large building of six or seven stories having a floor area of twenty to thirty thousand feet, panels approximately eighteen feet square, labor as outlined, sand at $1.00 per yard, cement, $1.20, crushed stone, $1.40, capacity of floors three hundred pounds per foot; rough slabs, columns and footings may be erected at an approximate cost to the contractor of about forty cents per square foot of floor area. Where the building is narrow and there are more columns in proportion to the floor area, on the same basis fifty cents per square foot would be a reasonable price.

Reduction in the floor load carried makes a relatively small reduction in the cost of the construction, since the centering would be the same for the light and the heavy building.

Where the load is increased fifty per cent above these requirements the additional cost would not be increased over eight per cent. While doubling the load would not increase the cost over about ten or eleven per cent.

This is the general type of information the shrewd contractor carefully figures out for himself and which enables him quickly and accurately to check up estimates made by his assistants or even to take work on an approximate estimate of this kind without going into details. Turner, when pressed for time once took a $60,000 contract on a twenty-minute estimate based on a computation only of the floor area and a knowledge of the conditions covering labor and cost of materials.

Where there are plain reinforced floors resting on walls and the panels are of large size such as in court house work and many other public buildings and where gravel can be cheaply obtained the cost per foot of floor may run as low as 30 cents per square foot. In other localities 45 cents per foot under less favorable conditions would be a reasonable figure.

In conclusion, it may be said that the prices of materials of construction vary from time to time. They differ in different localities and the same is true of wages and the efficiency of labor. General data on cost is therefore of value only as it presents an analysis of the items which enter into the unit cost. These items may then be changed to agree with the current prices, wages and efficiency of the class of labor for the locality of the work. More cannot be presented in a discussion intended to be of permanent rather than temporary utility.

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APPENDIX

By C. A. P. Turner

STATE OF THE ART OF REINFORCED CONCRETE FROM THE PATENT STANDPOINT AND THE MENACE TO PROGRESS BY UNSCIENTIFIC DECISIONS

In view of numerous unwarranted statements as to the state of the art, and right to royalty for use of various arrangements of reinforcement, a review of the state of the art relative to details of commonly used construction, and a brief history of their origin may be found of interest and value.

COLUMNS. In the early history of concrete column construction, columns were generally reinforced by longitudinal bars but with a great variety of ties. Hennebique used punched flats connecting vertical bars at the corners and occasionally there-between, making up a reinforcement of six or eight rods distributed in the outer portion of the column. Where the loads were heavy he occasionally used very large rods and a large percentage of steel which would not be considered good practice today.

The earliest appearance of the idea of spirally wound and vertically reinforced concrete columns is that disclosed in the 1874 patent to Hyatt. Hyatt's reinforcement consisted of a spiral of flat steel with vertical rods therein all placed as near the surface as proper fire protection permitted. Hyatt seems to have practiced his invention only to a very limited extent, since at that time little more was done in England and this country than to reinforce short span slabs and the like.

In 1885 Lee & Hodgson patented a reinforcement for a hollow concrete shaft consisting of reversed spirals riveted together at their intersection. British patent No. 14,559.

The hooped column was not exploited in the commercial field until after the experiments of M. Considère, had shown the great advantage of hooped columns vertically reinforced. Letters patent were granted to Considère in France and Belgium and an application for U. S. Patent was filed in 1902. Because of the limitations imposed by the prior art patent of Hyatt in England, 1874, and Lee & Hodgson 1885, the original claims of Considère were rejected, and a disclaimer was then filed limiting the interpretation of the claims in effect to the strengthening of a hardened core of concrete by winding coils of metal thereon. Considère describes his invention, as follows:

> "This invention has for its object to produce concrete or cement columns or the like structures which in service are adapted to withstand great compression or tension strains in a longitudinal direction. This object is secured by embedding strengthening-wires in helicoidal coils around a solid concrete core and covering the coils to protect them from the atmosphere."

Claims 1 and 4 are typical and are here quoted:

> "1. A column or the like of concrete, comprising a SOLID* concrete core with independent helicoidal coils of metal surrounding said core, and arranged very close together substantially as described.
>
> 4. A column or the like of concrete, comprisong a SOLID* concrete core, longitudinal rods within the core and independent helicoidal coils of metal arranged very close together and surrounding the core and rods and a covering for the coils, substantially as described."

---

*The significance of the word SOLID in these patent claims embodies the disclaimer differentiating the scope of the claim from the prior art of Hyatt, in which concrete was to be cast within and around the coils.

The prior art thus forestalled any claim by Considère upon the spirally reinforced column as customarily used in the art. The profession nevertheless is deeply indebted to him for his scientific work in introducing the safest and best type of concrete compression member in use today. The claims as modified by the disclaimer should entitle him to protection for the strengthening of old structures in which hardened cores are wound or bound with hooping and then covered or encased for protection. While this patent is in force the authors will respect it and advise their clients to do so and pay reasonably therefor in repair work where such mode of strengthening may be adopted. It might be maintained that in view of the prior patent of Hyatt, novelty could not be maintained in winding the hardened core. There might be just ground for this position, were it not a fact that the public were not using columns of this type and the benefit of the doubt should clearly be given that eminent pioneer who has done so much to reduce concrete column construction to the safest and most economical basis.

The principle of the hooped column is thus very old. The embedment of vertical reinforcement within the hooping is likewise old. All later patents therefore embodying the principle of the hooped column come within the classification of improvement patents limited in scope by the broad invention of the prior art.

Thus, improvement patents covering details of spacing devices for the coils or hoops, because the broad principle of hooping is old and the fastening of the spirals at some desired spacing is likewise old, are confined as treated by the Patent Office, to the specific improvement in means for accomplishing the desired result. The various patents issued to Clement, Cummings, Smith, Turner, Anderson, and others, upon spacing devices, are thus logically limited in the scope of their protection to the specific details originated by the respective patentees.

BEAM PATENTS. Since the pioneer patents of Scott, Monier, and the work of the earlier Monier licensees, improvement patents in beam reinforcement are limited as in the case of columns to details of construction, either to assemble them in a specific manner for convenient shop fabrication or handling, or to advantageous arrangement of bent up bars or to simplify the work of bending or to embody some real or supposed advantage in resistance to moment or shear. None of these patents, however, are broad enough to control construction or to prevent broad competition since the claims of all may be avoided by the use of older details which approximate if they do not equal the efficiency of the patented constructions. Fair business dealing would require regard of and respect for such improvement patents which appears somewhat lacking.

FLAT SLAB AND COLUMN CONSTRUCTION. The earliest appearance of the concrete floor flat in form is that found in the English patent granted to Scott, a Lieutenant Colonel of Engineers of the British Army, in 1867. This patent shows a flat concrete floor spanning from wall to wall, reinforced in the bottom throughout with hoop iron, in one direction in some sketches and in two directions in others and with fabric or wire netting embedded therein in the bottom of the slab in others.

The next patented floor flat in form is that of the 1875 English patent to Payne which shows various forms of groined arches upon separated piers with tie rods running directly and diagonally from pier to pier. In his specification Payne states that for light loads he would use a flat body of concrete with the reinforcements disclosed embedded therein. In other words, he would construct a typical flat arch reinforced lengthwise, crosswise and diagonally with steel tie rods therein.

In 1892, U. S. patent No. 467,141 was granted to Seely. This structure was arranged with small I shaped bars radiating in four or more directions from a cast iron column and embedded in the bottom of a thick slab with strap iron ties attached to the coulmn at the top of the slab and to the radial bars about one fifth of the span distant from the column, at the bottom of the slab, One building only was erected in accordance therewith.

A. Matrai, applied for patent in 1896, for a floor built upon the suspension system, designed to use the concrete as a filling material. Figure 1 shows the arrangement of wires on a rectangular frame. Quoting the specifications—

> "The frame is formed of girders b, supported at points 1, 2, 3 and 4; six series of wires being arranged parallel to the four sides of the rectangle and diagonally across the same."

Claim 1—

> "In a structure such as described the combination of a frame adapted to be supported at suitable points 1, 2, 3, 4, groups of wires extending from points in proximity to one of said points of support to points in proximity to another point of support and conforming to the shape of the bearing surface to be formed, and a filler in which said groups of wires are embedded, for the purposes set forth."

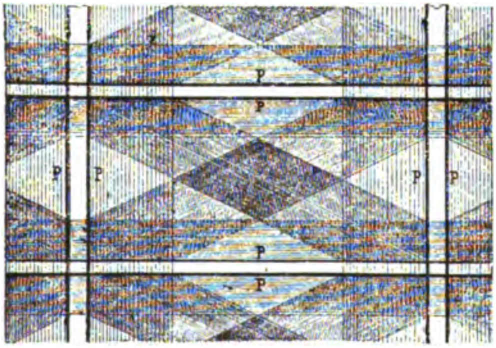


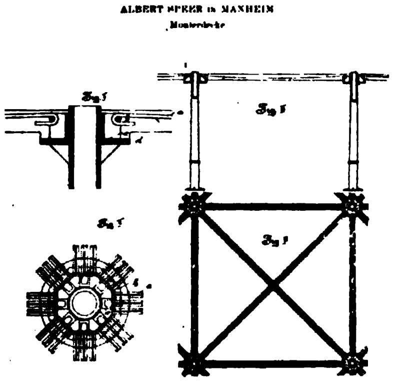
Illustration of the Matrai patent in "Ciment Arme"

According to Marsh, Herr Matrai's idea was to use the steel wires to support the load as in a swing, pulling against a steel frame work thereby avoiding any possibility of collapse due to uncured concrete, careless workmanship or poor materials. His idea was to use a cheap mixture such as lime concrete to spread the weight over the hammock of wires. The motive and principle was as far removed from that of reinforced concrete in which the compressive strength of concrete plays an equal part with the tensile resistance of the steel as it is possible to imagine, yet the apparent similarity of the structure to a reinforced concrete floor, caused one court, as will be noted later, to hold it an anticipation of a floor operating upon the basic principle of interdependence of the concrete and metal as in a true reinforced concrete structure.

IV

The next patent of interest is the Austrian patent to Albert Speer of Mannheim, applied for in August 1899, given out in October 10, 1900. Quoting the specification:

> "The present invention aims to get rid of girders and succeeds by connecting the four corners of the panel diagonally and direct by rods. Also these rods transmit the floor load direct to the columns Especially the diagonal connection is very important, because it supports the floor slab materially in the center and takes the greater part of the stress. The floor slab is suspended by this device at the four corners. In this way the girders—necessary until now—are quite left out, because it is clear, that with this construction only tension rods are necessary to transmit the load. On top of these rods is used a light two way metal mesh reinforcement, and the whole is filled in with concrete. Such a floor can have a flat ceiling, an advantage above all other constructions."



*The arrangement presented by the Speer patent is that of strips of rods lengthwise, crosswise and diagonally, passing over and under each other at mid span and hooked together at the column. The principle, like that of the Matrai floor, is that of a suspension system, but unlike the Matrai structure, it presents a column supported flat slab without projecting ribs. It differs from the continuous beam in that the steel in each strip is disposed in a catenary curve which resists the major portion of the shear at all points along its length and would therefore cooperate with the concrete much less than is the case in continuous beam construction. Because of the inequality in distribution of reinforcement through the slab the contours would not be circular but would be alternately convex and concave toward the center with lines of inflection at points of change of curvature at which there would be zero stress in a circular direction. Hence, in the Speer structure and in the Norcross floor also, there could be no cooperating resistances circularly and radially, a thing which is vital to the commercial feasibility of column supported flat slab floors.

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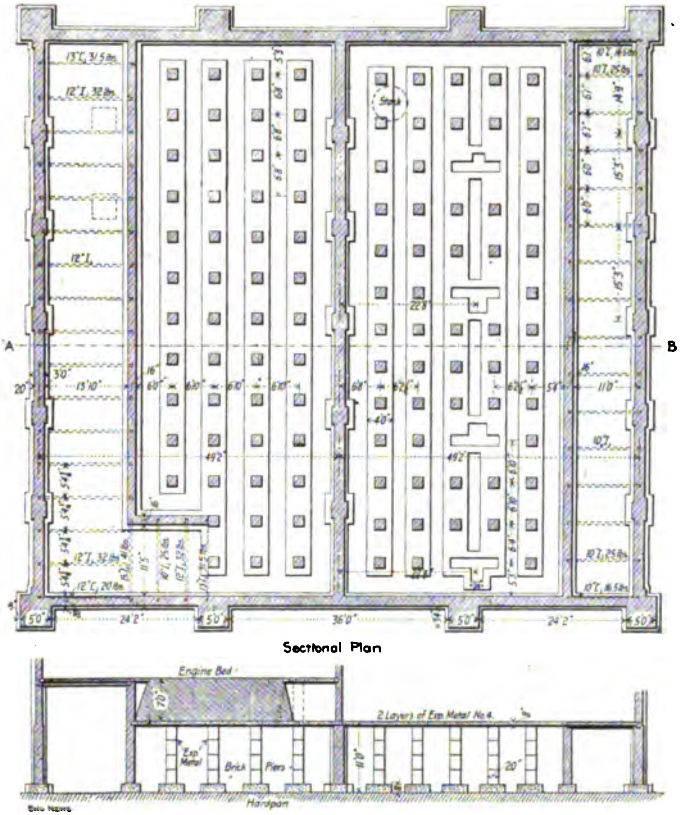
*This arrangement of strips lengthwise, crosswise and diagonally, over and under each other and over the top of the posts in a flat slab of uniform thickness covering the area of the floor, erroneously called a unitary panel by the court, was credited to Norcross by the Eighth C. C. A. in the case of Drum vs. Turner, through lack of familiarity with the prior art of record in the Patent Office.

V

In the Engineering News of September 14, 1899, the Far Rockaway Power House floor is illustrated. This was a flat slab floor, reinforced with expanded metal in the bottom throughout on separated closely spaced piers. Like the Payne structure, it was a flat arch, in which expanded metal was substituted for the steel tie rods of Payne.



Sectional Plan

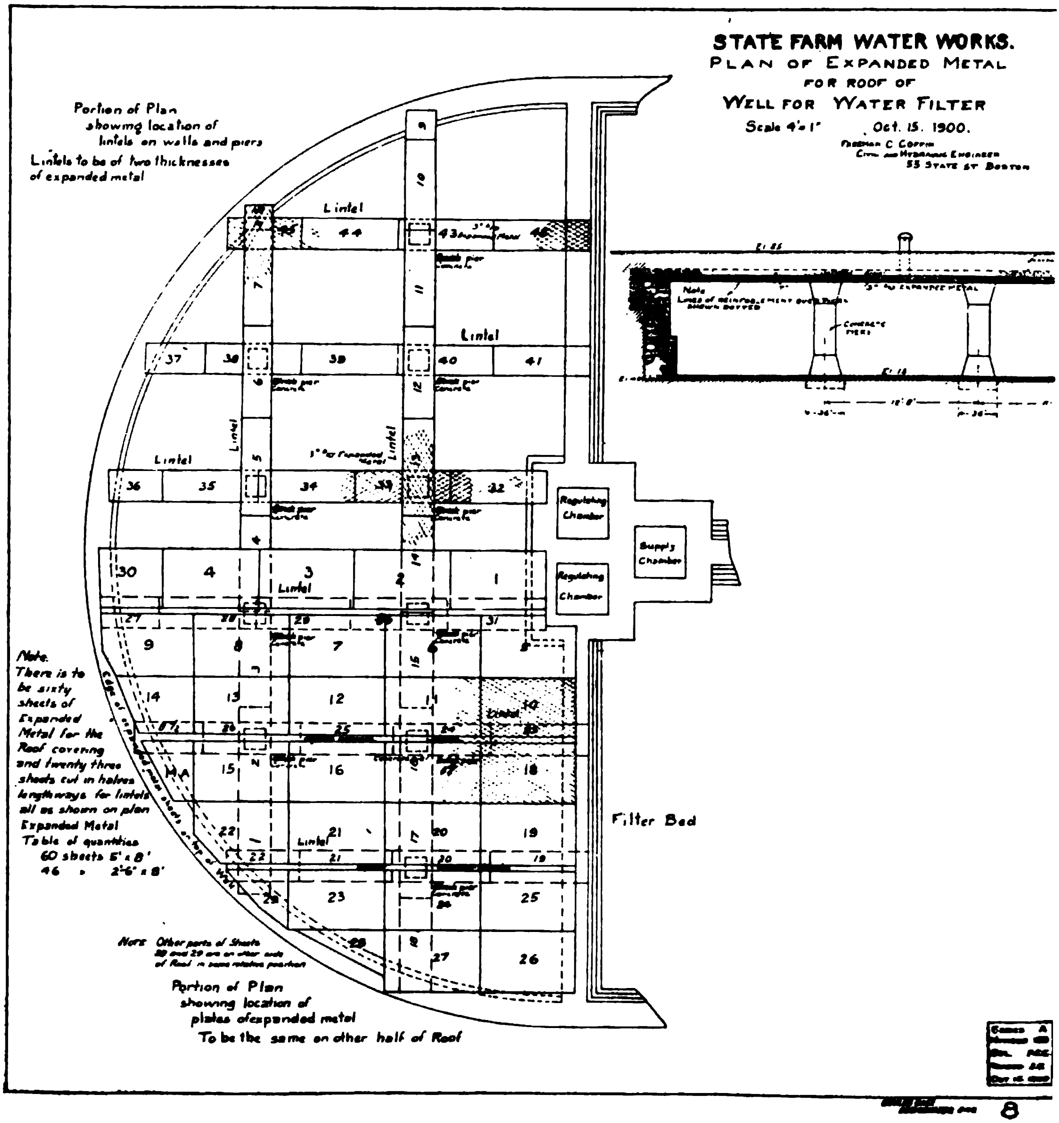
Transverse Section A-B.

FIG. 1.—PLAN AND SECTIONAL ELEVATION OF FOUNDATIONS AND FLOORS, QUEEN'S BOROUGH ELECTRIC LIGHT AND POWER CO.'S POWER-HOUSE, FAR ROCKAWAY, L. I.

October 15, 1900 plans, here reproduced were made for a Reservoir roof at the State Farm Water Works, Bridgewater, Mass. These plans were signed by Fredrick C. Coffin, and published in Folder No. 14 "Doings of Expanded Metal." It will be noted that the strips of expanded metal are marked "lintels" extending from column to column. The slab is of uniform thickness with expanded metal in the bottom extending in a single layer from lintel to lintel. In other words, this structure was a beam structure in action because of the concentration of metal in the lines of the lintel and because the reinforcement over the supports was arranged in the form of a cross which could resist stresses radially in two directions only, and could furnish no circumferential resistance about the supports at the line of inflection because there is no metal between the arms of the crossed sheets.

A more ambitious attempt in expanded metal construction was made by George Hill and described in the Railroad Gazette, January 3, 1902. The building was approximately 500 feet long by 51 feet wide. It consisted of a floor about

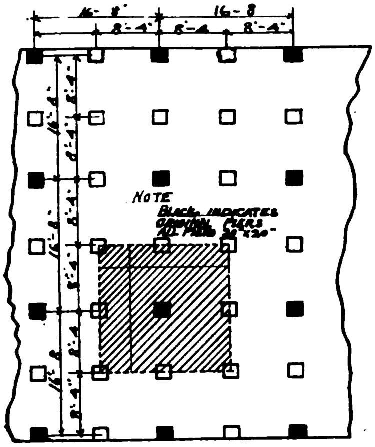
VI


STATE FARM WATER WORKS.
PLAN OF EXPANDED METAL FOR ROOF OF WELL FOR WATER FILTER
Scale 4'=1" Oct. 15. 1900.
Freeman C. Coffin Civil and Hydraulic Engineer 53 State St. Boston
Portion of Plan showing location of lintels on walls and piers
Lintels to be of two thicknesses of expanded metal
Lintel
Regulating Chamber
Supply Chamber
Filter Bed
Concrete piers
Note. There is to be sixty sheets of Expanded Metal for the Roof covering and twenty three sheets cut in halves lengthways for lintels all as shown on plan
Expanded Metal Table of quantities
60 sheets 5' x 8'
46 " 2'-6" x 8'
Note Other parts of Sheets 28 and 29 are on other side of Roof in same relative position
Portion of Plan showing location of plates of expanded metal
To be the same on other half of Roof
8

4'0" above the ground on piers 16'8" centers, and a flat slab roof supported on cast iron columns. It was designed to carry 400 pounds per square foot and to be used as a railroad storehouse carrying the usual supply of bolts, castings, brasses, nuts, washers, etc. It was represented by the defense in the suit of Turner vs. Lauter Piano Company as an early successful construction of flat slab floors carrying heavy loads. Computations based upon the obvious beam principle of action fixed by entire lack of reinforcement capable of resisting circular deformation in the cantilever portion, indicated that the structure could not possibly have carried the loads claimed. An investigation by crawling under the floor disclosed that the floor supports, instead of being piers 16'8" on centers as represented, comprised brick piers in addition to the concrete piers originally built, reducing the span to 7′0″ in each direction. The spans were thus so short that the 1′0″ thick slab arched from pier to pier and needed no reinforcement whatever.

Investigation brought out the statement by the storekeeper in charge that shortly after the structure was completed, failure of certain parts occurred under loads of less than twenty five tons per panel or less than 200 lbs. per square foot, and this in a floor intended to safely carry 400 lbs. After the collapse the brick piers were built as shown on the following sketch under all panels, furnishing nine points of support for each panel where four only had been contemplated in the original design.

An attempt was made at the same time the storehouse was built to construct a flat slab reservoir roof supported by columns spaced about 22′0″ centers. The slab was 8″ thick. Upon examining this Reservoir, the original concrete piers spaced 22′0″ on the original plans were found, and as in the storehouse, additional brick piers had been built under the slab at intervals. Seven concrete piers were designed to support the structure. The number of brick piers added was thirty one. The slab when first completed could not support its own weight; hence the liberal number of new piers added.



The failure of this experiment in the construction of flat slabs led to the abandonment of early attempts to build flat slab floors. It was no wonder in the light of such experience that the concensus of opinion of the engineering profession was (as stated by Capt. John S. Sewell, Trans. Am. Soc. C. E. 1906) that no economical system of concrete floors could be constructed without rolled steel beams or concrete T-beams.

NORCROSS PATENT. Late in 1901, O. W. Norcross applied for two patents on flat arch floors reinforced with hog wire netting, illustrated in the following figures. The specifications fixed the hog netting in the bottom and the claims of patent No. 698,542, were allowed by the Patent Office only after a disclaimer of improvement other than the advantage of strips of wire netting over rolled iron of all forms when used for flat arch reinforcement. The date of this application was subsequent by more than two years, to the printed description of the flat slabs in the Queensboro Electric Light & Power Station, Eng. News, Sept. 14, 1899; subsequent to the published Doings of Expanded Metal describing the Bridgewater Farm Reservoir, of 1900; and subsequent to the Austrian patent to Speer of 1900, disclosing six rolled round rods as direct and diagonal reinforcement in a column supported flat slab, all art that was prior to the Norcross application, but was avoided by the specific terms of a disclaimer.

Norcross thus describes his flooring, lines 10 to 15, page 2, specification U. S. Patent No. 698,542:

> "The metallic network of my flooring is formed by strips of suitable wire-netting. In practice I have used a hog wire fencing,

> which is a fencing of the same class as ordinary poultry wire fencing, except that the same is made of considerably heavier wire. In laying a flooring constructed according to my present invention, the posts are first erected and a temporary staging is built up level with the top of the posts." "Strips of wire netting are then laid loosely in place in the top of the staging.
>
> "In practice I have laid this (hog) wire netting in straight lines from post to post and crossing diagonally from post to post. A number of layers of this netting are preferably embedded such layers being laid together at crossing points in cob house fashion.... The concrete is then spread upon or moulded in place on the steel to enclose the metallic net work." (Lines 20 to 35 p. 2 Spec.)

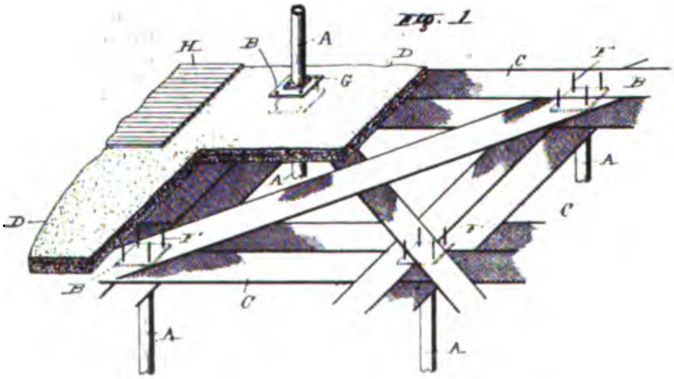
He analyzes his floor, as follows, lines 100 to 130 p. 2. Spec.

> "If the forces acting upon a section of the flooring supported between two posts be analyzed it will be found that the tendency of the floor section to sag between its supports will cause the lower layers of the floor to be under tension while the upper layers are under compression, these strains being of course greatest at the top and bottom layers... In addition to this the weight of a section causes a shearing strain at its line of contact with the supports. The principle upon which I have worked in constructing my flooring is to permit the concrete alone to resist compression and to supply a maximum amount of metal where the flooring is to be subjected to the greatest tension and shearing strains... The distribution of metal in any one span following **substantially the same rules that would be followed in distributing metal in the construction of flat arches.**"

The significance of the disclosure of the mode of operation becomes unmistakable when considered in the light of the Norcross second patent No. 698,543, the two patents being illustrated in parallel in the following cuts.

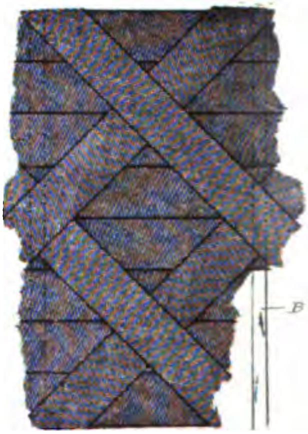
THE TWO NORCROSS FLAT ARCH FLOORINGS ILLUSTRATED

*U. S. Patent No. 698,542, Flooring for Buildings, O. W. Norcross. Filed Nov. 22, 1901.



Claim 5: A flooring resting on separated supports, consisting of concrete having a metallic network enclosed in the bottom layer thereof, with the body portion of said concrete of lighter material than the bottom layer thereof.

U. S. Patent No. 698,543, Flooring for Buildings, O. W. Norcross. Filed Dec. 23, 1901.



Claim 2: The combination of walls or other longitudinal supports, and a flooring consisting of a panel of concrete having a metallic network inclosed in the bottom layer, said metallic network comprising strips of wire netting arranged crosswise with respect to each other.

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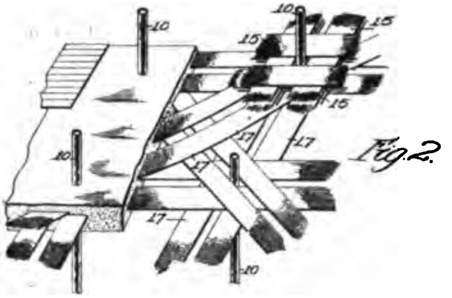
*This patent expires April 29th, 1919, but claims for royalty and threats to sue are still being made.

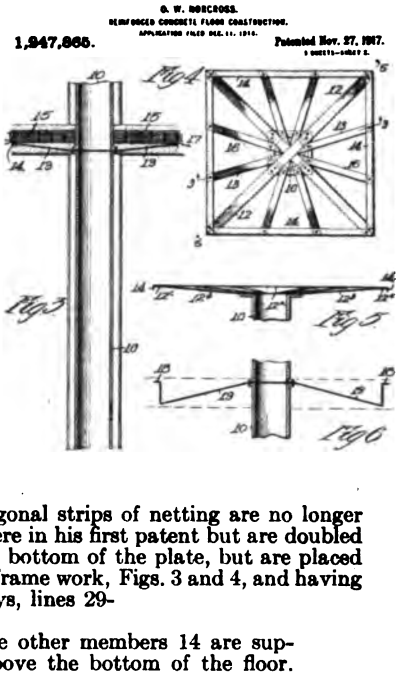
Quoting lines 70 to 76 patent No. 698,543, to Norcross:

> "The especial object of my present invention is to provide a modified form of flooring constructed on the same principle as in my previous application for patent 698,542, but which instead of being supported on separated columns, or supports, is designed to rest directly upon walls or other longitudinal supports."

It is thus established beyond a doubt that if Norcross had any idea of a principle other than that of a flat arch he did not disclose it to the public at that time.

December 11, 1916, Norcross applied for his third flat slab patent issued November 27, 1917, patent No. 1,247,865. Figures 2, 7, 8, 9 of that patent are reproduced below.





It will be noted in Fig. 2 that the diagonal strips of netting are no longer half the width of the direct strips as they were in his first patent but are doubled in width and that they are no longer at the bottom of the plate, but are placed at the top of the plate and supported by the frame work, Figs. 3 and 4, and having members 14. Of these the specification says, lines 29-

> "It will be observed... that the other members 14 are supported at a considerable distance above the bottom of the floor. On these members several thicknesses of flat metal reinforcements 15 are located... These are in the form of rods or preferably woven wire cloth... They do not pass directly over the columns as in my above specified patent 698,542, but pass them on each side. Additional continuous tie members 17 are shown, several radiating from the columns. This gives an opportunity to allow them to hang down between the columns as indicated in Figs. 1 and 2... In this way they exert their strength not exactly in horizontal direction but upwardly so that the floor is supported at a distance from the columns by slanting reinforcements which exert an upward pull. **This is of great importance over constructions in which the pull is nearly or substantially horizontal.**"

This specification is of particular interest in view of the strenuous effort by speculators to make the public at large believe that Norcross invented this very thing in the year 1902, a matter which will be referred to later.

Norcross' original claims as filed for his first patent 698,542, were rejected upon the patent to Seely, which showed small I-shaped bars in four directions radiating from the columns which combined with like bottom position of the metal therein caused the rejection of the original Norcross claims as appears from the Patent Office letter of December 23, and claims as quoted:

Patent Office Letter

Dec. 23, 1901.

This case has been examined.

The 1st, 2nd and 3rd claims are met in the patent to Seely, 467,141, Jan. 12, 1892 (Concrete and Metal).

The 8th claim is for a mere temporary combination of flooring and tool, but presents nothing of invention over Seely above who shows the metal structure on posts and the ordinary cement filling put in with centering in the usual way.

The 9th claim is for a separate and independent improvement in centering, which is a distinct art, and said claim cannot be covered in the old generic combination: See ex parte Herr O. G. 41 page 464.

The 1st, 2nd, 3rd and 8th claims are rejected and the 9th is objected to.

Original Claims As Filed

1. A flooring consisting of concrete having metallic net-work enclosed therein so as to radiate from the posts on which the flooring rests.

2. In a building, the combination of posts or other separated supports, and a flooring resting on said supports, and consisting of a single panel of concrete having disconnected metallic net-work enclosed therein.

3. A flooring resting on separated posts, and consisting of concrete with a plurality of layers of metallic net-work enclosed herein.

4. A flooring resting on separated supports, and consisting of concrete with metallic net-work so arranged therein that the amount of metal will be greatest at the points where the greatest tensile and shearing strains are to be supported.

5. A flooring resting on separated posts, and consisting of metallic net-work formed by strips of wire netting laid from post to post to cross each other in cob-house fashion, and concrete inclosing the metallic net-work.

6. A flooring resting on separated posts, and consisting of metallic net-work formed by strips of wire netting laid from post to post, and on the diagonals of the figures outlined by the posts, and concrete enclosing the metallic net-work.

7. A flooring resting on separated supports, consisting of concrete having a metallic net-work enclosed in the bottom layer thereof, with the body portion of said concrete of lighter material than the bottom layer thereof.

8. The combination of a concrete flooring resting on separated posts, and temporary props for supporting the flooring at intermediate points after the removal of the stagings.

9. The combination of a concrete flooring resting on separated posts, and temporary spring-props for supporting the flooring at intermediate points after the removal of the staging, said props comprising top and bottom pieces loosely connected by a rod or pipe so that the top and bottom pieces can be driven into or out of place in setting or removing said props.

Amendment:

Feb. 24, 1902.

In reply to the office letter of Dec. 23, 1901, we desire to amend this case as follows:

Please concel claims 1, 2 and 3, without prejudice, and substitute therefor:

1. "The combination of separated posts or supports, and a flooring consisting of metallic net-work formed by strips of wire netting enclosed therein, so as to radiate from the posts or supports on which the floor rests."

Cancel claims 8 and 9, without prejudice.

Renumber the claims.

A reconsideration is requested.

The patent of record to Seely shows a construction in which rolled-iron plates and beams are enclosed in concrete.

One especial object of the applicant's invention is to dispense with the use of rolled-iron of all forms, employing wire netting in place thereof.

The claims rejected, however, have been erased, and the single claim substituted therefor has been drawn especially with a view of bringing out this fact, so that as now presented, the case is thought to be in condition for allowance.

By the disclaimer of rolled iron of all forms, round as well as plates or shapes, Norcross was allowed claims by the Patent Office, differentiating from the flat slab floor of Speer with round rods, by specifically claiming strips of wire netting lengthwise, crosswise and diagonally, radiating from the posts, on the applicant's contention that wire netting in these directions was more efficient than all forms of rolled iron as a tie. Thus the Norcross patent as granted by the Patent Office was limited to an almost microscopic scope. It was later expanded by the courts to a breadth to cover the prior art in a manner undreamed of by the patentee, and considered possible only by an enterprising speculator.

This was done by the following language of the Eighth C. C. A. decision, Drum vs. Turner:

> "But this statement (of the disclaimer) must be read and given effect in the light surrounding the circumstances at the time he made it, in the light of the question which was before him when he made it and in the light of the knowledge then possessed of the use of rolled iron for the purpose for which he used his strips of netting."

But round rods had been disclosed in strips of six rods in the four directions in a column supported flat slab in the Speer patent, and the light in which this disclaimer was interpreted by the Patent Office was in the light of Speer, in the light of Payne, in the light of the use of round rods generally in reinforced concrete and the disclaimer of round iron of all forms was not in the light of sole prior use of rolled I-beams only but in the light of the use of rods in place thereof, and of wires in four directions as in Matrai.

Norcross made two attempts to use his construction and then only when building upon a percentage basis, the one a floor in the Wightman House at Brooklyn, Mass., approximately 25′ by 35′ on walls with two brick piers dividing this area into six panels of a little less than 11′ clear span with a slab a foot thick; the other in the Coolidge Residence at Manchester, Mass. This floor was about 44′ by 26′ in total area supported on the outer edge by heavy walls and by three brick piers dividing the floor into eight panels of approximately 11′ by 12′0″ in size and a total floor thickness of 15″ including 3″ marble tile. The netting found therein was placed in the bottom throughout, crosswise from wall to wall, covering the area, and with additional bridging strips across the piers according to Norcross' patent 698,543, making it a three way reinforcement. The mesh was 4″ by 6″ hexagon mesh like poultry netting with wires of the size of stovepipe wire, samples of which taken from the floor had less strength than a shoe string, because they were of black wire rusted before embedment. The great thickness of the slab and its manner of support made it stable as an arch of plain concrete. Netting of such small cross section added nothing to the strength of the floor.

All efforts subsequent to 1902 to construct flat arches such as Norcross patent, or concrete plates reinforced with expanded metal and supported in the manner of the Hill structures were abandoned because such structures were less economical than ordinary beam and slab types.

Mushroom Floors. Early in 1898, Turner was experimenting on the thermo electric measurement of stress and planned a series of experiments on concrete plates. Business engagements delayed the proposed experiments. In 1903 plans for a structure similar to the early type of Mushroom floor were made for a flat building for a Minneapolis architect, but these plans were never executed.

An effort was made to incorporate the new idea of construction in the roof of the Minneapolis Paper Company building in 1905, but it was vigorously opposed by the Building Department on the ground that it was experimental and hence necessarily untrustworthy.

A sketch of the proposed mat arrangement appeared in the Engineering News of October 1905, and later was illustrated in the Proceedings of the American Society of Civel Engineers in February 1906, in the discussion of a paper by Capt. John S. Sewell, on the economical design of reinforced concrete floors.

The first modern flat slab was the Bovey-Johnson building in Minneapolis, panels approximately 14′ by 17′, tested to 800 pounds per square foot over two adjacent panels. Following this structure the system was rapidly introduced in many large buildings as will appear from the following references to engineering publications:

Reinforced Concrete Conctruction, (259pp)
By Turneaure and Maurer, University of Wisconsin
Wiley, N. Y., 1907.

Concrete, Plain and Reinforced, (483pp)
By Taylor and Thompson
Wiley, N. Y., 1911.

Trans. Am. Soc. C. E.
Vol. LVI. June 1906.

Engineering News:—
Oct. 4, 1906, p. 361.
Feb. 18, 1909, p. 176.
Dec. 23, 1909, p. 694.

Engineering Record:—
March 28, 1908, p. 374.
May 2, 1908, p. 575.
Oct. 10, 1909, p. 411.
April 3, 1909, p. 408.
April 10, 1909, p 492.

The first building of this new type in Chicago was the Curtis Building on Jackson Boulevard, Francis M. Barton, architect, described by him in the Cement Era, in the fall of 1906. This building was hardly complete before it was advertised as the Barton system.

In Milwaukee the John Hoffman Sons Company building, on Water Street, was completed in 1907, and attracted considerable attention when tested with 1000 pounds per foot of floor. Mr. Daniel B. Danielson of Milwaukee was one of the early visitors to this building in January and February and in the following May applied for a patent for a column and flat slab construction which was similar to that in the Hoffman building except in the formation of the carrying rods on top of the bent out column rods and in place of the usual hooping for the columns, angular ties were claimed. Mr. Danielson later, erected some flat slab floors calling them "Musculous System". And so the "Systems" multiplied rapidly.

In St. Paul, Mr. D. C. Shepard, one of the pioneer engineers of the Northwest, was much interested in the description of the system in the Proceedings of the Am. Soc. C. E. in 1906, and visited the test of the Bovey Building in Minneapolis in the fall of that year. Satisfied with the construction he secured its adoption for a National Bank Building in St. Paul and proposed using it in a building which he contemplated erecting for himself.

After some study, Mr. Shepard's view was that the concrete plate would present greater toughness if also reinforced normal to the saddle back area and this view was met by incorporating in the design of two large buildings subsequently erected for him in St. Paul, reinforcements in two directions covering the area of the floor normal to column lines in addition to the diagonal belts customarily used. The first building, the Strong-Warner Building, was erected in 1909, plans being made the later part of 1908 or the first part of 1909. Preliminary negotiations for this building had been entered into in 1907, during the construction of the Lindeke-Warner building.

As Mr. Shepard preferred deformed bar reinforcement, corrugated bars were used in this building and cuts and photographs of the reinforcement were advertised shortly thereafter in the Engineering News by the Corrugated Bar Company, who afterwards advertised the flat slabs adopting Mr. Shepard's idea of reinforcement normal to the saddle back each way but thinning the mat by omission of the diagonal belts.

While Mr. Shepard, undoubtedly anticipated others in incorporation of reinforcement normal to the saddle back as a modification of the details of the mat as practiced by Turner, several patents have been issued thereon to-wit, Number 1,005,756 filed December 1909 long after the incorporation of this feature in the building named, and a later application filed April 24, 1911, eventuating in U. S. Patent No. 1,050,477 of January 14, 1913.

Turner Patents. Application for the original Turner patents was made in January 1905. The case was divided by the Patent Office two years later and the continued applications were vigorously prosecuted.

In prosecuting these cases interference proceedings were declared first with an application filed by Messrs. *Durkee & Prince, covering a modified form of flat slab, which used punched plates with radial rods hooked therein over the columns but lacked the characteristics of the mat. The claims, however, were drawn like the Turner claims and an interference was declared by the Patent Office on fifteen counts. Messrs. Durkee and Prince of course were aware that no priority could be established by them and disposed of their interest, Durkee selling his half interest for one hundred fifty dollars. This was purchased by parties unfriendly to Turner and the contest drawn out in the Patent Office as far as legal red tape would permit. Decision, as was to be expected was in favor of Turner.

Through an over-sight in the Patent Office a joint interference was not declared as it should have been at the time with an application by Francis M. Barton, when the Durkee Prince interference was brought on. Barton, as before noted was a Chicago Architect, who adopted the Mushroom system as he put the matter in an article written by him for the Cement Era, in 1906. This interference proceeding, like the other, was delayed as much as possible by Barton and his attorneys but it was finally brought to hearing on a motion to dissolve and decided in favor of Turner. Testimony was taken as regards priority but was of such a nature that Mr. Barton did not submit it and the decision as to priority was allowed to go by default after putting Turner to the expense of taking testimony and proving his rights.

The early introduction of the new construction was a somewhat strenuous business. Nothing was known of plate action in flat slabs prior to the data which Turner had secured regarding the action of concrete flat slabs so reinforced as to imitate plate action. The problem of this design was attacked in the manner outlined in Chapter II by the principle of proportion but every different ratio of depth to span and of column rigidity, which differed from the exact proportion from which the data was secured, brought in new questions and new problems and these had to be solved one at a time by careful observation and repeated tests.

The engineering profession then knew so little about this type of construction that Prof. A. N. Talbot having been called upon to pass upon the design of The John Deere Plow Company building, Omaha, Neb., (see Engineering News, February 18, 1909, for description) reported to the owners after investigating a number of buildings erected on this plan, that he was unable to figure it. That he was satisfied it could be made stable, and he advised the owners to require Turner to put up a $50,000 bond guaranteeing the load capacity and deflection under load. Turner furnished the bond, guaranteeing the structure at the request of Mr. C. M. Leonard, President of the Leonard Construction Company, contractors for the building. The floor was tested satisfactorily after the concrete had become thoroughly cured.

Almost every construction in the early days required some kind of a guarantee, either from the contractor doing business with Turner or from Turner himself. Some of these bonds reached the magnitude of $100,000 as in the case of the Y. M. C. A. Building at Vancouver, B. C.

It would seem, if the patent system were at all efficient, that after such strenuous work in introducing so new a system of construction, shouldering the financial burden of heavy guarantee as just noted, and spending months in the most careful investigations, that the difficulties of the pioneer should have been ended. They had in fact, however, only commenced. Data borrowed and tables of capacity appropriated likewise enabled early imitation. The open disclosure made by the publication in 1909 of methods of computation, through misplaced confidence in the protection of a United States patent, also tended toward the same result.

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*Builders of the Bovey building from Turner plans.

About the year 1910, C. M. Leonard, of the Leonard Construction Company, who built from Turner plans the John Deere building described in Eng. New. Feb. 18, 1909, *bought of O. W. Norcross, patent No. *698,542 placed the title to the same in the name of an employee,† one John L. Drum, and proceeded to erect flat slab constructions indentical with the Turner floors except in the minor details of the spider, but the same in principle and operation. Turner brought suit against clients of the Leonard Construction Company for infringement and the ‡Leonard Construction Company brought suit against Turner, in the name of Drum alleging infringement of the Norcross flat arch patent by the construction of the continuous plate patented by him.

Moore Scriver Suit. About the same time suit was brought by Turner against a Minneapolis infringement, in the building of Moore & Scriver, designed by an engineer, who had become familiar with the Turner construction by handling plans during his former connection with the Building Department of the City of Minneapolis. While the structure was as nearly like the Turner structure as two peas from the same pod, the trial court held no infringement. The Court of appeals of the Eighth Circuit held that flat slabs were old in the art and so declared by the Patent Office, and so of their reinforcement in various directions over the supports; citing Ciment Armè illustration of the Matrai construction, as reference, and held the claims sued upon invalid as an aggregation. (211 Fed. 466).

Drum vs. Turner. The Drum vs. Turner decision followed the Turner vs. Moore in the Eighth C. C. A. 219 Fed. 199, in which the court said, (page 190):

> "The principle or mode of operation of the construction of the floors which are alleged to infringe the patent of Norcross, is disclosed in the drawings and specifications of Turner's patent Nos. 985,119 and 1,003,384."

And held this allegation a legal fact in a decision which was antipodal to its previous decision, Turner vs. Moore & Scriver and that of the Patent Office in Barton vs. Turner as well.

The examiner, skilled in his art, based his finding on his knowledge of what had been accomplished in the past. The court, being without such knowledge, held that the disclosure of the arch, ancient as it is in principle, is broadly patentable and that a patent thereon is infringed by a structure which is not an arch in any sense of the word.

The antipodal nature of the decision of the Engineer Examiner and the Court, appear in their diverse holdings upon the question of invention, improvement, scope of claims, utility (success vs. failure) and finally upon mode of operation upon which infringement depends. These contradictions appear clearly from the quotations in parallel columns as follows:

Drum vs. Turner Decision Eighth C. C. A. holding Turner patent No. 1,003,384 an infringement of patent 698,542.	**Patent Office Decision Barton vs. Turner giving an engineering analysis of the prior art and mode of operation of Turner and Norcross structure.**
"Nor have the contentions that the flooring of Turner is not the mechanical equivalent of that of Norcross, because	Discussing the patentability of the issue, it is said of the entire prior art, to which attention has been called, that it

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*Norcross statement to F. H. Drury.

†Having learned from Turner how to successfully build column supported flat slabs, the inconsistency of suing Turner as an infringer of a paper patent was avoided under cover of the name of an employee. In the accounting Drum was replaced by the Flat Slab Patents Co. in which by the statement filed by its counsel about as much of the stock is owned by Leonard as law permits an individual to own and still operate under the legal alias of a corporate name.

‡Testimony of Prof. W. K. Hatt as to the identity of his paymaster in the case.

the cross wires in the strips of netting have an effect in keeping the long wires from slipping which is not produced by the belts of small rods, but the belts of small rods are spaced apart and tied to keep them in place in the actual construction of Turner's device, because Turner places his rods in the upper while Norcross places his wires in the lower part of the flooring where they lie over the tops of the columns, because Turner's belts of rods are wider than Norcross' strips of wire-netting, because Turner has enlarged the capitals of his columns and constructed cantilever heads for them and for other reasons of like character, failed to receive consideration. Conceding that these changes which Turner claims to have made were improvements on the device patented to Norcross, they did not deprive the plaintiff of his right to remedy for the defendant's appropriation and use by equivalent means of the principle of his invention. The answer to all of these contentions is that the basic principle or mode of operation of the defendant's construction of beamless flooring, is strips or belts of large wires or small rods embedded in concrete and laid over the tops of columns in the way described in and secured to Norcross by his patent, whereby a monolithic floor enclosing the wires or rods in a single panel, however large the building, is made to sustain itself and its load on separate columns, without girders or beams. Extract this principle or mode of operation from the devices of Turner, and they become inefficient and useless. The floors will not sustain themselves on the posts. And as long as he continues to use the principle of Norcross and the mechanical means which he described and secured, or their mechanical equivalents, he cannot escape infringement.

The review of the prior art, the long and fruitless endeavor before the disclosure of this patent to find a way to construct concrete floors for large buildings without beams or girders, the gradual approach to the desideratum, its first conception and reduction to practice by Norcross, and its large and successful subsequent use, have satisfied that his flooring and his method of construction were not within the reach of the mechanic skilled in the art, but were novel and useful and the product of the genius of the inventor."

fails utterly to disclose or to even suggest the subject-matter of the invention included in this interference, and that the only references that are worth while to reconsider at this time are the patents to Matrai, Speer, Seeley, Norcross and Folder No. 14, and this only for the reason that these references have been given especial weight in this motion to dissolve.

In applying any reference to this invention, it is to be borne in mind that Turner has achieved that which engineers and practical men have regarded as impossible, that which has met with instant approval and success wherever the structure has become known and that which the moving papers of Barton concede is not to be anticipated except by piecing references together and raising the objection of "no invention."

The patents to Matrai, Speer and Seeley are all simply modifications of the suspended type of floor construction. Any resemblance found in these structures to the invention involved in this interference is believed to be accidental only, more apparent than real. The vital and essential features of the Turner construction are entirely lacking. The mode of operation and the results attained by Turner would be impossible with any of these structures alone or taken in combination with the structure disclosed in the patent to Norcross or the Folder No. 14.

Referring particularly to the Norcross patent, it is a vital thing in securing the results attained by Turner to have the cantilever head arrangement at the top of the columns and the slab reinforcement extending therefrom in multiple directions through the slab, or, as defined in varying terms in the issue count, whereby a cantilever suspended arrangement results and the combined cantilever and flat plate action is secured.

In Norcross the strips of wire netting are plainly intended as they are plainly shown and described, lines 21 and 32, page 2 of the specification, as mere supports at the bottom of the slab of concrete placed thereon. Clearly the reinforcement of Norcross is of the ordinary kind and is relied upon only as a tension element in the lower chord. As a cantilever reinforcement in the position shown at the bottom of the slab, it would be an entire failure.

But in Drum vs. Turner the holding that patent 985,119 is the same in principle as Norcross the court reversed its previous holding, 211 Fed. 466, that

Turner's "flat slab was old, and so of its reinforcement by groups of rods passing in various directions over the support" in the light of the illustration in Ciment Armè, of the Matrai structure disclosed in U. S. Patent No. 634,986 of 1899, applied for in 1896, five years prior to the Norcross application.

The first bench apparently had difficulty in reading the mechanical drawings of this publication. The plan view of Matrai's flooring is strikingly similar to that of patent 985,119. The cross section and elevation, however, show clearly the ribs or beams therein.

In Drum vs. Turner the Court of Appeals was influenced by the same similarity of plan view found in the later patent to Norcross. It apparently applied the legal metaphysics of change of degree to the Norcross patent to arrive at a conclusion of identity of principle between the structure of Turner and that of Norcross as argued by counsel.

As few engineers are conversant with the adaptability of this legal line of reasoning, its remarkable possibilities so useful to the patent pirate may be advantageously illustrated. In the design of beams every one knows that they are made deeper or shallower dependent on questions of economy on the one hand or the desired head room on the other. The difference between the deeper and shallower beams is thus a change in degree and not a change in principle, but if we make the change in depth such that the beams of the Matrai floor become of the same depth as the slab we have a floor structure flat on top and bottom in which according to the court's finding in Drum vs. Turner, there could be no beams' concealed or otherwise, since it would be obviously flat on top and bottom. Thus by a change in degree the Matrai floor became in legal fiction a flat slab such as the Turner flat slab and such as the Norcross flat slab and by thus applying this adaptible line of reasoning, things which are different become in legal fiction the same. Matrai becomes the same in principle as Norcross and the same as Turner. This same adaptbale principle was applied in argument to the Norcross netting. If it was in the bottom it would be but a change of degree to put it in the top. If it did not cover the area so that there was uniform flexibility it would be again but a change of degree to widen it out so that the flexibility would be uniform. The application of this doctrine involves no consideration whatever of the radical change in mode of operation which may be brought about by an apparently insignificant change in detail. The entire legal argument rests upon the assumed non-existance of such change in mode of operation.

Thus the Turner flooring, imitating the continuous homogeneous plate, was held first to be old in the light of an illustration resembling it merely in plan which was not a flat plate at all, and again only a year later, a like resemblance to a flat arch (so described by the patent specification) was considered ground to hold Turner's novel floor an infringement of the very same elements which the same court had just held were old, and when assembled constituted in the fiction of the law an aggregation in accordance with the Hailes vs. Van Wormer decision.

Thus under the same state of facts, the same question is answered by the same court of appeals affirmatively and negatively against the same litigant in different cases dependent solely on the wide or narrow scrape of the imagination of the particular bench applying in an unscientific manner the doctrine of change in degree.

The need of competent scientific ability in the correct determination of questions relating to mechanical patents is thus emphasized.

Lack of understanding of specifications leads not only to inequitable but to inharmonious decisions in different circuits as appears from a comparison of the decisions of Drum vs. Turner and of Turner vs. Lauter Piano Company, which for comparison, are quoted in parallel columns, as follows:

Drum vs. Turner, 1914, Eighth C. C. A. Patent in suit No. 698,542.	Turner vs. Lauter Piano Co., 1918, Third C. C. A. Patent in Suit No. 1,003,384.
Norcross patent held valid and infringed although the specification asserts: (Lines 115-121 page 2 of the specification) "The principle on which I have worked in constructing my flooring is... to supply the maximum amount of metal at points where the floors are subjected to the greatest tensions and shearing strains." (Line 105) "The tendency of a floor section to sag between supports will cause the lower layer of the flooring to be under tension while the upper layers of the flooring are un- X **der compression, these strains being greatest at the top and bottom layers respectively.**" While the court held that the review of the prior art, the long and fruitless endeavor before the disclosure of this patent to find a way to construct floors for large build- Y ings without beams and girders, the gradual approach to the desideratum its first conception and reduction to practice by Norcross, its large and successful subsequent use, have satisfied that his floor and his method of construction were not within the reach of a mechanic skilled in the art but were novel and useful and the product of the genius of the inventor."	"Reinforced concrete of a modern kind came in to use... about fifty years ago... It is a matter of general knowledge that concrete is strong in resisting compression strains and weak in withstanding tension strains. Places at which both strains may be expected are so well known that they are determined empirically by many engaged in the art. There is today neither invention **or novelty in merely placing metal reinforcement in concrete at places where these strains come.**" X' "The vital thing which the patentee claims to have achieved... is the making of a thin slab flooring which will carry as heavy load as the thick slab flooring of the prior art at a lesser cost. "There doubtless is merit in Turner's system of concrete construction. It may be superior to others but merit may spring from a conception that does not involve invention. These qualities come from a careful assemblage of different elements from various sources and clever combinations of them." Y'

Antipodal rules of law are here illustrated by holding that invention consists X in putting the steel where the strains are alleged to come in the Eighth C. C. A., and that X' it is not invention to put the steel where the court thinks (?) strains come in the Third C. C. A. Large and successful *use (Y) satisfied the Eight C. C. A., as to invention but in the Third Circuit on the contrary, wide introduction of a structure admittedly superior to others involves merely clever combination of different elements producing economy and superiority but not invention. (Y')

Pretended extensive commercial use of the original thought of a paper patent on an abandoned experiment, decked out in present day details vital to commercial success admittedly discovered not by the patentee but by the defendant is accepted as sufficient commercial use of the patented thing itself to sustain invention and infringement of a paper patent in the Eight Circuit while on the contrary actual and extraordinarily wide commercial use and conceded superiority is insufficient in the Third Circuit to sustain patentable invention.

The obviously antipodal holdings upon the same state of facts apparently arise from the acceptance of supposed discovery in the one circuit and its rejection in the other as a valid basis of invention. But the antipodal character of these decisions is exhibited in other respects.

The Norcross patent sued upon in Drum vs. Turner, sets forth the principle, lines 125 to 130 p. 2 of the specification, as "distributing the metal following the rule of distributing metal in flat arches"—an old rule dating back to the time of the Roman Empire. Now assembling elements all old according to this ancient principle constitutes invention in the Eight C. C. A. decision as quoted while in the Third Circuit we have the holding—

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*In the accounting it was conceded under cross-examination that plaintiff had never built, used or tested the structure disclosed by the Norcross patent, 698,542.

> "Invention is to be found only in discovering a new principle or employing a new means involving an old principle."

The position of these two circuit courts of appeal is again antipodal. And finally as if to render the inconsistency complete the Third C. C. A. quotes with approval the finding of the Eighth C. C. A. as a substantial premise from which by legal logic to deduce its antipodal conclusion and holding of legal fact.

Legal Facts vs. Mechanical Facts. Whenever the holding or decree of any court is inharmonious with, antipodal to or contradictory of scientific fact or mathematical truth, under the fiction of the law this holding is a legal fact binding upon the litigants **but not upon the public.**

This phase of the law renders it essential if industry is to be encouraged at all by the Patent system, that those who pass upon the validity of patents should have a clear understanding of scientific questions relating thereto. The statutes that have been enacted properly require that the examiners in chief of the Patent Office possess competent legal knowledge combined with scientific ability but competent scientific ability is not required of the judge who passes upon the validity of the patent. **Thus the final step which renders the system beneficial on the one hand or a menace to progress on the other, is taken haphazard without the guidance of competent scientific ability.**

Because scientific ability is lacking in their determination many decisions are mosaics of legal fictions, The Drum vs. Turner decision is a classic of such fiction. In it, every mechanical principle, mathematically demonstrated in the preceding chapters, is inharmonious with or contradictory to the legal facts therein contained. Even Norcross, himself, did not know that he had invented a continuous plate until the court so held, as appears conclusively from the sworn specification of his third patent. Of course the speculative purchaser of the Norcross patent, had ample testimony that the Norcross structure had been practiced by changing it to the modern continuous plate structure, and as a fitting reward for the ingenuity of the patent expert under our system of determining legal facts the pioneer inventor is duly punished for his discovery and the profits awarded to the enterprising speculator who knew where to purchase the right kind of testimony to convince a court that a flat arch was the same as a continuous plate; that putting the steel in the top of a floor was the plain mechanical equivalent of putting it in the bottom.

As long as courts, lacking scientific ability, reverse the scientific opinion of trained patent office examiers, they will continue to err and their decisions will continue to be a menace instead of an encouragement to progress. The patent speculator will continue as now to wax fat by the purchase of a paper patent on any commercial failure that bears a remote resemblance to any present day success. By bringing suit upon such paper patents, and pursuading the court that the relic which he has dug from the graveyard of commercial failure in fact contains the original thought of some revolutionary success he secures a decree in his favor and proceeds to blackmail the public in a wholesale manner. By this process the inventor who discovered the secret of success which was in no wise suggested by the abandoned experiements of the prior art is robbed of the reward for his discovery and penalyzed upon the legal fiction that his present day success infringes some prior art failure.

Lack of familiarity with the principles of equity is not the usual source of erroneous decisions. The difficulty lies in the inability of the court to formulate the actual state of facts in relation to the art to which the law is to be applied.

The first difficulty the Third C. C. A. encountered in the Lauter case was in understanding that a floor was really elastic—that it would bend, or deflect under a given load. When satisfied regarding this, an elementary explanation of the mode of operation was attempted by the theory of work to show that the elements acted in fact as a mechanism. While they signified an affirmative view on the question of the applicability of mechanical law of conservation of energy, they, nevertheless, erred in holding as a legal fact that all that Turner had done was to distribute the steel in all radial directions, as had in effect been done in the prior art. The idea that circular and radial combined could mean something different from and more than radial only was wholly beyond their conception of geometrical principles.

On the other hand, their finding that invention is to be found only in discovery of a new principle or in employing a new means involving an old principle was logical. Had it been followed in the Eight Circuit, the Drum vs. Truner decision would not have held that the disclosure of the flat arch principle was the essence of a continuous plate because the flat arch principle dates back to the time of the Caesars, as all students of engineering know.

Notwithstanding the diversity of the Eighth C. C. A. and the Third C. C. A. decisions, the same fundamental error is at the root of both, i. e., the idea that the concrete somehow fixes the position where the "strains" (in the sense of maximum stresses) come. No idea of the fundamental principle of rigidities is disclosed by either decision though explained in the testimony of both cases. Hence it could not have been expected that either court would recognize a new principle in the concrete floor even when exhibited before it.

Legal Facts vs. Historical Fact. The state of the art is determined by the history of that which has been accomplished in that art in the past. The court, lacking in scientific training is likewise lacking in this historical background from which alone improvement can be properly affirmed, or denied. The unprincipled attorney, aware of this weakness, fabricates an artificial historical back-ground to fit the avowed object of "putting his case across" or "putting it over the court". The result of this is that legal facts are rarely harmonious with historical facts. In the plaintiff's brief in the case of Drum vs. Turner, there occurs repeatedly the statement that at the time Norcross conceived his flat floor steel beams were universally used and therefore his disclaimer was the disclaimer of rolled steel beams because by implication, rods in place of beams had not come into use. The historical fact of the matter is that round rods were embedded in flat arches two thousand years prior to the invention of a rolling mill for the production of rolled beams or plates. An honest court, under the false impression that membership in the Bar is synonymous with integrity, gives credence to the misstatement of historical fact, unsubstantiated by any evidence in the record and inharmonious with the prior art exhibits as to the date and period, without careful examination to determine whether such argument rests upon fiction or fact.

As an American the author takes a justifiable pride in American institutions in general, but the repeated substitution of legal fictions in place of mechanical and historical facts in court decisions relating to patents is a discredit to an advanced industrial nation such as the U. S. A. It has been held that

> "No patentable invention can be practically or fairly understood if the language of the claims is entirely dissociated from the specification, and the claims and specifications should be read together." Washer Co., et al v. Cramer, et al 169 F. 629.

Further, that

> "The specification of a patent which forms a part of the same application as its claims must be read and considered with the latter, not for the purpose of limiting or contracting the claims but for the purpose of ascertaining their true meaning and the intention of the parties when they were allowed." O. H. Jewell Filter Co. v. Jackson, 140 F. 340.

And, again,

> "In the construction of patents and in the application of the law to the facts it presents words must be given the same meaning that the patentees gave them and must be used to designate the same things that they used them to designate or nothing but confusion and mystification can result." Drum vs. Turner 219. Fed.

Yet this latter court, after stating the correct principle, gave the terms flat arch, and mathematical flat plate, not the meaning the respective patentees Norcross and Turner gave to them in their patent specification, but employed them with no regard for the thickness of the plate necessary for arch action on the one hand or the permissible thinness on the other hand made possible by the arrangement of metal in two tension zones in mat formation to imitate the action of the homogeneous plate.

The fundamental relations which determine the mode of operation, economic feasibility and carrying capacity for a given amount of material in a slab, were lost sight of entirely in a superficial consideration of mere flat top and bottom surfaces found in the stone slabs of the ancient Egyptian and Greek temples as well as in present day artificial stone flooring of concrete and metal.

When Norcross used the term flat arch in his specification he meant (as a mason would understand it) something thick enough relative to its span to act as an arch, i. e., to contain within itself a curve of sufficient rise or height necessary to sustain itself and its *proof load by arch action although the external form was "flat".

When Turner used the term "mathematical flat plate" he meant (as any mathematician would readily see) a plate too thin in proportion to its span length to support itself and its guaranteed * proof load by arch action with the steel ties disclosed by Norcross in the position required by the old and well known rules for distributing metal in flat arches, lines 125 to 130 p. 2 specification patent 698,542. By this confusion of terms the matter was surrounded with that fog of uncertainty or mystification that the court held should be avoided, and thus was evolved the legal fiction that the ancient principle of the flat arch was identical with the new principle of imitation in a non-homogeneous plate of the mode of operation of the homogeneous plate. Thus was evolved that confusion of ideas in the legal fiction that putting the steel in the top of the floor was the plain mechanical equivalent of putting it in the bottom; that the ties of a flat arch would operate as efficiently in the top of the arch as when in the bottom, that mere plan direction of the metal was a controlling factor in mechanical action; that the width of the reinforcement, its uniformity of distribution or lack of uniformity had nothing to do with the case, and yet all of these elementary questions were set in an exact, clear light in the judicial opinion of the Engineer Examiner rendered in an inter partes interference contest in the Patent Office and placed before the court without avail for its enlightement and guidance.

There is a great difference in the use of terms, however, as employed by various callings. For example, nefarious occupations, such as that of the sneak thief and the yegg have a vocabulary all their own. The explosive which the engineer calls nitrogylcerine the yegg calls "soup". Where the inventor speaks of winning a patent suit, the legal sneak thief or shyster patent lawyer speaks of "putting it across" or "putting it over", an expression fitting the state of facts better than the word winning as applied to the favorable outcome of a patent suit brought about by his efforts. The expression "putting it over" is not a particularly respectful manner of referring to the mental acumen of the honest judge, learned only in the law, but it is quite expressive of the game that is played upon him by the shyster lawyer aided by the camouflage furnished in wholesale quantities by that business branch of the legal fraternity, known as patent experts.

The ease with which a court, lacking competent scientific ability, may be misled regarding fundamental mechanical principles, is not surprising when we consider that these principles are not apparent on the face of things but in fact

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*Proof Load, is defined, Rankine, Civil Engineering, p. 223

"If the piece is to be afterwards used, the testing load must be so limited that there shall be no possibility of its impairing the strength of the piece; that is, it must not exceed the proof strength, being one-third to one-half the ultimate strength. About double the working load is in general sufficient."

are hidden and do not appear plausible to those untrained in that branch of learning. But little over three centuries ago it was generally thought that the sun revolved about the Earth, and a court then so held against Gallileo, and enjoined this eminent philosopher from the publication of his scientific work setting forth the truth of the matter and from further teaching it and by this injunction the court robbed him of the royalties or copyright fees thereon, his sole means of support in his old age. In the course of the centuries which have elasped since the days of Gallileo, it would seem that little or no improvement has been made in methods of determining legal facts so that they harmonise with mechinical fact.

While the State Legislatures have made appropriate laws to safeguard ordinary property rights, no Federal Legislation has as yet made adequate provisions to render the property rights in inventions or in industries based on invented things secure.

The true meaning and legal sufficiency of the specification as defined by the highest authority and the statutes giving the true meaning force and effect in interpreting the claims arc irreconcilable and incompatible with each other. Thus the true meaning and legal sufficiency of the specification has been defined by the early English decision in Arkwright vs. Nightingale (C. P. 1775) as follows:

> "The specification is addressed to persons in the profession having skill in the subject and not to men of ignorance and if it is understood by persons whose business leads them to be conversant with such subjects it is a sufficient specification under the law."

This test of the sufficiency of the specification has been affirmed under our system in the holding of the Supreme Court in Carnegie v. Cambria, 185 U. S. 403, 46 L. Ed: 968:

> **"The patent is not addressed to lawyers, or even to the public generally, but the manufacturers of steel, [illegible] any description which is sufficient to apprise them in the language of the art of the definite feature of the invention... is sufficiently definite to sustain the patent."**

Nevertheless it is a fact that questions of validity and infringement depending on the true meaning of the claims as interpreted in the light of the specification is left by our statute to be decided by judges who are not persons having skill in the subject matter and conversant with it.

Now if the sufficiency of a specification to make it valid under the law be such that though it is not understood by those whose business has not led them to be conversant with the subject matter as it the case of the judges, their failure to understand it would obviously result in inharmonious views and the application of diverse rules of law to the same state of facts because of the difference in view point taken. Where the decision arrived at is based on the uncertain character of the best understanding the judge could arrive at, while *inefficient to pass on the questions at issue, the mathematical probability that two independent courts could possibly render harmonious and logically consistent decisions is so remote that the mathematical probability in favor of their conflict amounts to almost a certainty.

Again, in cases where the courts through ignorance of the subject matter inadvertently and unintentionally determine legal facts contradictory of and inharmonious with natural laws, the probability is equally great that two different courts will write opinions inharmonious and conflicting with each other.

The conditions above discussed are thus summarized by the committee of the Patent Law Association of Chicago, in its report on the Oldfield. Bill

> **"That Patent Litigation is Expensive, the Results Uncertain, and Decisions Conflicting Must be Admitted."**

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*See Equity Cases 152-153 Circuit Court Beaumont Eastern Div. Texas.

The remedy is obvious†: the enactment of a statute requiring that the question of the validity of a United States patent shall be passed upon by a court possessing competent legal knowledge and scientific ability to understand the subject matter, just the same as as is required by the existing statute as a qualification of Examiners in Chief in the Patent Office, to enable them to pass upon the allowance of patent grants. The system would then become harmonious and not be such a house divided against itself, that an unscientific court reverses the patent office expert on ten or fifteen scientific questions in one and the same decision.

Since the foregoing was written, a decision has been handed down by the Eighth C. C. A. in Luten vs. Washburn, et al, quoting with approval, Judge Hook's opinion in Turner vs. Deere & Webber Building Company, 249 Fed. 753, regarding the use of concrete in building construction, as follows:

> "Its resistance to compression and susceptibility to other stresses in certain positions were familiar to all who had to do with it, as were the general principles of reinforcing it with wire, rods, or strips of metal which possessed the quality the concrete lacked. When the plaintiff entered, the art had so progressed that the nature of the stresses and in a general way the places where the reinforcement should be disposed or arranged were a part of the common knowledge of builders, as in greater degree was the subject of struts, braces, and the like in carpentry. In pretentious or complicated construction, where ordinary experience did not suffice, mathematical computation was available. The evidence of prior practice in building and prior publications and patents show that little was left for patentable invention in placing the customary pieces of metal here or there, or turning them this way or that, in the mass of concrete."

Quoting the Court:

> "In Drum v. Turner, 219 Fed. 188, 135 C. C. A. 74, we sustained the Norcross patent for a metallic concrete flooring without supporting beams. The invention in that case, however, was for a floor without supporting beams, not merely the location of the reinforcing rods."

The strange decision in Drum vs. Turner that a flat arch, old in principle, is the equivalent of a mechanically continuous plate, was thus reached from the false premise that the invention of the patent in suit was for a floor without supporting beams. The opinion treated all slabs FLAT IN FORM, supported on piers or columns when reinforced by netting instead of shapes, as mechanical equivalents alike in static resistance without regard to the area of distribution of metal or even lack of it, in one or both of the two tension zones. A study of the construction of the ‡Model Power House, (described in the Engineering News Publication September 14, 1899) built and used more than two years prior to the Norcross application, shows that any structure thus broadly defined by the court's opinion was old in the art and under the literal provisions of the patent statute, unpatentable.

The fact that the Court, the same Judge presiding in both cases, in the Luten case thus explains the Drum case, appeared to the writer to raise a Constitutional question and the matter was referred to an eminent patent lawyer for an opinion regarding it, and his reply is quoted as follows:

> "I have your esteemed favor, suggesting that there is a Constitutional infirmity in the decision of Drum vs. Turner, 219 Fed. Rep. 188, and inviting comment on the suggestion.

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†Notwithstanding the obvious nature of the remedy its application is strenuously opposed by the majority of the fraternity of patent lawyers because they mistakenly believe they can by the continuance of the present system profit more in cash, than the value of the esteem and respect of their fellow citizens.

‡Reproduced p. V of this discusion.

> In reaching the decision referred to, the Circuit Court of Appeals, if credit may be given to the explanation thereof lately given by the same court in Luten v. Washburn, 253 Fed. Rep. 950, erroneously held or assumed that the Norcross invention, at the filing date of the Norcross patent application, was new and patentable notwithstanding the fact that a printed description of the same invention had been published more than two years before that filing date in the Engineering News of September 14, 1899. This is understood to be the point which you suggest. If you are rightly understood, therefore, the infirmity of the decision amounts to no more than this, namely, that the court erred in holding the Norcross patent valid regardless of the statutory antiquity of the invention. This was outrageous, but not apparently unconstitutional, for this court has a constitutional right to make blunders of all kinds, subject only to [*] correction, in patent cases, by certiorari from the Supreme Court."

This constitutional right of the Courts of Appeal to err in scientific matters has been freely exercised relative to reinforced concrete as shown by more than one expression of judicial opinion that little is left for patentable invention in placing customary pieces of metal here and there and turning them this way or that, since doing this very thing under the principle demonstrated in Chapter IX radically changes the mode of operation, the stiffness and load carrying capacity of the structure in a most astonishing manner. The discovery of how to produce a novel success by placing the metal this way or that, therefore may involve an entirely new principle and bring the subject matter under the correct holding of principle by the Third C. C. A., that invention in reinforced concrete is to be found in discovering a new principle or employing new means embodying an old principle. The labors of the latter court altho based upon a correct rule, brought forth twin errors since it was unable to recognize the new principle of imitation of plate action in concrete slabs or the new means, that of a mat covering the area of synclastic curvature embodying the old principle of strengthening the concrete by steel, because unfamiliar with and unversed in elementary mechanics.

The unsatisfactory character of the provisions for the determination of scientific fact in patent causes, raises the question of responsibility therefor. This responsibility rests solely upon the United States Congress. Were the erudition in scientific matters of the average Congressman no higher than that of the Phillipine Igorrote, such a system would be explicable. But notwithstanding the fact that nearly all members of Congress are lawyers, this is supposedly not the case. The existence of such a system is therefore quite inexplicable and for an industrial nation predominantly Caucasian, like the U. S. A., it is not only inexplicable, but highly discreditable as well.

The culpable indifference of Congressmen to the encouragement of industrial progress of the country is evidenced by the fact that in the space of fifty years it has enacted no commensurate statutory increase in the pay of Patent Office employees, notwithstanding the fact that the increased cost of living during that period in the estimate of these same Congressmen fully warranted more than doubling their own compensation.

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[*] That the precise disclosure of plan, elevation and section is commonly misconstrued and misunderstood by the courts in patent cases is not a matter of surprise to those mechanically trained. Such natural error might be corrected by the present procedure of the petition for rehearing, did it not appear probable, as the lawyers commonly advise clients, that such petitions are seldom read and never greanted for mechanical errors. Congress has not provided that the court deciding the equities of a patent case shall have competent scientific ability, and a court lacking such ability, having determined the issue as best it can, that is the end of it as far as that court is concerned.

The probability of correction by certiorari is equally remote. A misconstruction of the exact disclosure of the drawing appears, under our faulty legal system, to be regarded not as a matter of fact but as a mere matter of opinion regarding the evidence and involving no question of law or equity recognized under the present system. It is only theoretical technicalities of the law that are rigorously reviewed and the errors therein carefully corrected by superior courts, and thus gross errors and miscarriages of justice are rarely corrected as thousands of pages of the Federal Reports bear witness.

Notwithstanding that judicial opinion in equity renders that so in law, which is not so in fact, such errors commonly involve inharmonious application of the rules of law. Thus in the Drum case, as explained in the Luten case, the holding of patentability of the form of a flat floor which determines in part the form of a room, flatly contradicts the holding in the case of the Folding Bed Co. vs. Anselsteen, Ninth Circuit, in which the form of the room was specifically held unpatentable. Thus the rule of law which applies to John Doe in the Ninth Circuit does not apply to Richard Roe in the Eight Circuit, and the contrarieties of our unscientific system of patent equity is disclosed.

Such extreme indifference to the business welfare of the country has not been exhibited by the Federal Judiciary which has made some limited improvement in old methods by new rules under which scientific testimony is taken before the court instead of before a notary, and the straightforward manner in which the technical witness meets techincal questions may thus be more accurately adjudged than from mere printed records. But this change in procedure in no wise compensates for the handicap of lack of technical understanding of the subject matter by the court.

With the rapid advance in the scientific arts, the lack of competent scientific ability on the part of the court is becoming such a serious matter that the frequency of irrational determinations of the equities of patent suits is increasing by leaps and bounds.

In the consideration of invention vs. design there is the greatest variety of opinion. Some of the editors of technical papers have taken the position that a thing that is designed could not be invention. What more in fact was the successful conquest of the air by the Wright Brothers than the scientific design of wings, steering apparatus and the application thereto of the gasoline motor not invented by them† In a recent editorial the statement occurs—

> *"Arrangement of reinforcement to meet stresses is now fairly definitely established as mere engineering design and not invention."

But if that is established today in accordance with beam theory it was also established in 1906 when the concensus of opinion of the leading members of the engineering profession was that no extensive system of concrete floors could be economically designed without rolled steel beams or concrete T-beams. There was therefore no sufficient suggestion in the earlier efforts to enable an engineer skilled in structural design to adapt the principle to meet acceptible theory. And on the question of acceptible theory the failure of building codes and committees on reinforced concrete in the past twenty years to develop any rational theory of design to cover the principle of the mat discovered by Turner is a sufficient commentary upon the error of such unscientific view of the situation.

Such ex post facto wisdom is defined by the Supreme Court rule that:—

> "Knowledge after the event is always easy, and problems once solved present no difficulties, indeed, may be represented as never having any, and expert witnesses may be brought forward to show that the new thing which seemed to have eluded the search of the world was always ready at hand and easy to be seen by a merely skilful attention. But the law has other tests of invention than subtle conjectures of what might have been seen and yet was not. It regards a change as evidence of novelty, the acceptance and utility of change as a further evidence, even as demonstration". Diamond Rubber v. Concolidated, 200 U. S. 428, 55 L. Ed. 527; 31 S. Ct 444.

Invention is admittedly difficult to define but the rule as to practical success expressed by the Supreme Court in the Telephone case tends to clarify the matter:

> "It was left for Bell to discover that the intermittent current could not be made under any circumstance to reproduce the vibrations of the human voice but the true way was to operate on an unbroken current by increasing and diminishing its intensity.

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*Eng. News Editorial April 24, 1919.

†Maxim designed the first heavier than air machine to rise from the ground.

> This Reis never thought of and he failed to transmit speech. Bell did and he succeeded. Under such circumstances it is impossible to hold what Reis did was an anticipation of the discovery of Bell. To follow Reis is to fail. To follow Bell is to succeed." Telephone Cases 126 U. S.; 1 31 L. Ed. 863; 8 S. Ct. 778.

The test of efficiency came into notice in the case of Loom v. Higgins, 26 L. Ed 1177. In that case, altho the patent disclosed no single marked improvement but a series of minor improvements which taken together showed increased efficiency of the carpet loom of from 40 yards per day in the hands of a skilled workman to 50 yards per day in the hands of an unskilled workman. Loom Co. Higgins. 105 U. S. 580; 26 L. Ed. 1177.

The rule regarding marked utility as applied to invention is found in the case of O'Rourke v. McMillan, 160 Fed. 933; 88 C. C. A. 115, as follows:

> "Has the patentee added anything of value to the sum of human knowledge† Has he made the world's work cheaper, easier, safer, would the return to the prior art be a retrogression† When the court has answered this question, or these questions in the affirmative, the effort should be made to give the inventor the just reward for the contribution he has made. The effort should increase in proportion as the contribution is valuable. Where the court has to deal with a device which has achieved undisputed success and accomplished a result never attained before, which is new, useful and in large demand, it is generally safe to conclude that the man who made it is an inventor.

The law laid down broadly for the construction of patents by the Supreme Court in Wyman vs. Denmead, 15 Howe, V. 41, is that—

> "Patents should be construed liberally in accordance with the design of the Constitution and the patent laws of the United States to promote the progress of the useful arts and allow inventors to retain to their own use not anything which is a matter of common right but what they themselves have created."

These general principles expressing the fixed law of patents as stated by the highest authority, may serve as a guide to the reader in forming an independent opinion upon the question whether the field of reinforced concrete should be considered as some would like to see it almost the only branch of human endeavor in which regardless of the intent of the Constitution there is no power to promote the progress of its science as a liberal art by securing for a limited time to inventors the exclusive right to their inventions and discoveries.

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